POWERS OF COMPLEX NUMBERS AND DE MOIVRE'S FORMULA

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If we write a complex number of unit magnitude in exponential form, we have

$$z = e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

Raising this to a power, we have

$$(\cos\theta + i\sin\theta)^n = e^{in\theta} \tag{2}$$

$$= \cos(n\theta) + i\sin(n\theta) \tag{3}$$

This is De Moivre's formula:

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)} \tag{4}$$

Although n is usually taken to be an integer, nothing in the derivation assumes this, so the result is also valid for an n of any value (even complex).

A common use of De Moivre's formula is in finding expressions for multiple angle trig functions.

Example 1. Find $\cos 3\theta$ and $\sin 3\theta$.

We have, using the binomial theorem

$$\cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3 \tag{5}$$

$$= \cos^3\theta - 3\sin^2\theta\cos\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right) \tag{6}$$

Taking real and imaginary parts

$$\cos 3\theta = \Re\left[(\cos\theta + i\sin\theta)^3\right] \tag{7}$$

$$= \cos^3\theta - 3\sin^2\theta\cos\theta \tag{8}$$

$$\sin 3\theta = \Im\left[(\cos\theta + i\sin\theta)^3\right] \tag{9}$$

$$= 3\cos^2\theta\sin\theta - \sin^3\theta \tag{10}$$

Example 2. Powers of sin and cos can be easier to calculate using exponentials. For example,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{11}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{12}$$

From this, we have

$$\sin^4\theta = \frac{1}{(2i)^4}\left(e^{i\theta} - e^{-i\theta}\right)^4 \tag{13}$$

$$= \frac{1}{16}\left[e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta}\right] \tag{14}$$

$$= \frac{1}{16}(6 + 2\cos 4\theta - 4\cos 2\theta) \tag{15}$$

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