PRODUCT RULE AND INTEGRATION BY PARTS

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We'll have a look at two common techniques in calculus that are not usually considered together. The reason we do so here is because they are essentially inverses of each other.

The first is the rule for calculating the derivative of the product of two functions, such as $h(x) = x^2 \sin x$. We can derive the general formula using the definition of the derivative

$$h(x) = f(x)g(x) \tag{1}$$

$$\frac{dh}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \tag{2}$$

We can work out the limit on the right by the little trick of adding and subtracting the same quantity in the numerator.

$$\frac{dh}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \tag{3}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \tag{4}$$

$$= \lim_{\Delta x \to 0} \left[g(x+\Delta x)\frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x)\frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \tag{5}$$

$$= \frac{df(x)}{dx} g(x) + f(x)\frac{dg(x)}{dx} \tag{6}$$

where we have used the definition of the derivative twice, and the fact that $\lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$ to get the last line.

Thus the product rule states (using the prime notation for derivatives: $df/dx \equiv f'$):

$$(fg)' = f'g + fg' \tag{7}$$

As a simple example, the derivative of the function above is

$$h' = \frac{d(x^2)}{dx}\sin x + x^2\frac{d(\sin x)}{dx} \tag{8}$$
$$= 2x\sin x + x^2\cos x \tag{9}$$

The product rule for derivatives can be used to derive a common method of integrating the product of two functions, known as *integration by parts*. To derive the rule we can integrate the product rule formula:

$$\int (fg)'\,dx = \int [f'g + fg']\,dx \tag{10}$$
$$fg = \int f'g\,dx + \int fg'\,dx \tag{11}$$
$$\int f'g\,dx = fg - \int fg'\,dx \tag{12}$$

That is, if we can identify in the integrand (the term being integrated) a product of two functions, at least one of which we know how to integrate, we can identify one term in this product as the *derivative* of f and the other term as the function g.

For example, suppose we want to integrate $h(x) = x\sin x$. We can identify this as a product of two functions, and then let:

$$f' = \sin x \tag{13}$$
$$g = x \tag{14}$$

We then get

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)\,dx \tag{15}$$
$$= -x\cos x + \sin x + C \tag{16}$$

where C is as usual the constant of integration. The formula is easily checked by taking its derivative, using the product rule derived above.

In using integration by parts it is important to choose the right way of splitting up the product of two functions in the integrand. In the above example, if we had chosen $f' = x$ and $g = \cos x$ then applying the integration by parts formula would just make the situation worse (try it and see - you'll end up trying to find $\int x^2\sin x$).

Integration by parts sometimes allows the use of an even more devious trick that works with a few functions. The most common example found in introductory calculus courses is in finding the integral of the logarithm. On the face of it, there is no obvious function whose *derivative* is $\ln x$ so it seems we're stuck for an integral. However:

$$\int \ln x \, dx = \int (1)(\ln x) \, dx \tag{17}$$

$$= x \ln x - \int (x) \left(\frac{1}{x}\right) dx \tag{18}$$

$$= x \ln x - \int (1) \, dx \tag{19}$$

$$= x \ln x - x + C \tag{20}$$

That is, sometimes we can identify the constant 1 as the derivative f' in the integration by parts formula. With the logarithm, this works since the integral of 1 cancels out the derivative of $\ln x$ nicely to give an easy integral at the end.

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