

## PROJECTION OPERATORS - FORMAL TREATMENT

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Continuing from our examination of orthonormal bases and the orthogonal complement in a vector space  $V$ , we can now look at the orthogonal projection, sometimes known in physics as a projection operator.

Suppose we have defined a subspace  $U$  of  $V$  and its orthogonal complement  $U^\perp$ , so that  $V = U \oplus U^\perp$ . We can define a linear operator  $P_U$  called the *orthogonal projection* operator. It has the property that, given any vector  $v \in V$ , it 'projects' out the component of  $v$  that lies in  $U$ . That is, if we write

$$v = u + w \tag{1}$$

where  $u \in U$  and  $w \in U^\perp$ , then

$$P_U v = u \tag{2}$$

In what follows, we will usually refer to  $P_U$  as just a 'projection operator', although a projection operator in physics almost always refers to an *orthogonal* projection operator.

An example of a projection operator is an operator in 3-d space that projects a vector onto the  $xy$  plane. Then the  $xy$  plane is the subspace  $U$  and the  $z$  axis is the orthogonal complement  $U^\perp$ .

From the definition of  $P_U$  we can list a few properties:

- (1)  $P_U$  is not surjective, that is, its range is smaller than the entire space  $V$ .
- (2)  $P_U$  is not injective, since it maps all vectors  $u + w$  to  $u$ , for all  $w \in U^\perp$ . Thus it is a many-to-one mapping.
- (3)  $P_U$  is not invertible, since it is not injective. Another way of looking at this is to consider a projection in the real world where we use a flashlight to project the image of some 3-dimensional object onto a plane, creating a shadow. We cannot tell from the shadow alone what type of 3-dim object was projected. For example, if the shadow is a circle, the 3-dim object could be a sphere, cylinder, cone or any other object that could cast a circular shadow if illuminated from the right angle.

- (4) Its null space is  $\text{null } P_U = U^\perp$ .
- (5) Once  $P_U$  is applied to any vector  $v$ , all subsequent applications of  $P_U$  have no effect. That is, once you've projected out the component of  $v$  that lies in  $U$ , all further projections into  $U$  just give the same result. In other words  $P_U^n = P_U$  for all integers  $n > 0$ .
- (6)  $|P_U v| \leq |v|$ . That is, the 'length' of a projection must be less than or equal to the length of the original vector. This follows from the Pythagorean theorem, since  $u$  and  $w$  are orthogonal, so  $|v|^2 = |u|^2 + |w|^2 \geq |u|^2 = |P_U v|^2$ . Geometrically, a projection operator cannot increase the 'length' (norm) of a vector. This property relies on the fact that the projection is an orthogonal projection. Other projections can increase the length of a vector (think of the shadow cast by a stick; if the surface onto which the shadow falls is nearly parallel to the direction of the incoming light, the shadow is much longer than the stick).

An explicit form for  $P_U v$  can be obtained from the decomposition we had earlier

$$v = \underbrace{\sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U^\perp} \quad (3)$$

From this,

$$P_U v = \sum_{i=1}^n \langle e_i, v \rangle e_i \quad (4)$$

From the definition, it seems reasonable that a vector space  $V$  can be decomposed into a direct sum of range  $P_U$  and null  $P_U$ . We can in fact prove this.

**Theorem 1.**  *$P$  is an orthogonal projection within the vector space  $V$  if*

$$V = \text{null } P \oplus \text{range } P \quad (5)$$

*Proof.* We can take the subspace  $U = \text{range } P$ . From our earlier theorem, we know that  $V = U \oplus U^\perp$ , so we need to show that  $U^\perp = \text{null } P$ . Since  $Pw = 0$  for any  $w \in U^\perp$ , then  $\text{null } P \subset U^\perp$ , but are there vectors in  $U^\perp$  that are not in  $\text{null } P$ ? Suppose there is such a vector  $x \in U^\perp$  such that  $Px \neq 0$ . For such a vector, we can decompose it into  $x = x' + x''$  where  $x' \in \text{null } P$  and  $x'' \in \text{range } P$ , with  $x'' \neq 0$  (since if  $x'' = 0$ , then  $x$  would be in  $\text{null } P$ , contrary to our assumption).

As  $x \in U^\perp$ ,  $\langle x, u \rangle = 0$  for all  $u \in U = \text{range } P$ . Therefore  $\langle x, u \rangle = \langle x' + x'', u \rangle = \langle x', u \rangle + \langle x'', u \rangle = 0$ . Since  $x' \in \text{null } P$ ,  $\langle x', u \rangle = 0$  (as  $x' \in$

$U^\perp$ ). Therefore we must have  $\langle x'', u \rangle = 0$ , implying that  $x'' \in U^\perp$  also. Thus  $x'' \in U$  and  $x'' \in U^\perp$ , but the only vector that can be in both a subspace and its orthogonal complement is 0, so  $x'' = 0$ , which contradicts our assumption above.  $\square$

From property 5 above, we must have  $P_U^2 = P_U$ , which implies that the eigenvalues of  $P_U$  are 0 and 1. The eigenvectors belong to either the subspace  $U$  (for eigenvalue 1) or to the orthogonal complement  $U^\perp$  (for eigenvalue 0).

The orthonormal basis of a vector space  $V$  can be divided into two separate lists of vectors, with one list  $(e_1, \dots, e_m)$  spanning the subspace  $U$  and the other list  $(f_1, \dots, f_k)$  spanning  $U^\perp$ . A matrix representation of  $P_U$  can be obtained by considering the action of  $P_U$  on each of the basis vectors from the two subspaces. We have

$$P_U e_i = e_i \quad (6)$$

$$P_U f_i = 0 \quad (7)$$

In general, the matrix representation of an operator  $T$  is defined in terms of its action on the basis vectors  $v_i$  by

$$v'_j = \sum_{i=1}^n T_{ij} v_i \quad (8)$$

For a projection operator, we can see that this means that for the  $m$  basis vectors  $(e_1, \dots, e_m)$  we must have  $P_{ij} = \delta_{ij}$  for all  $i, j = 1, \dots, m$ , while for the  $k$  basis vectors  $(f_1, \dots, f_k)$  we must have  $P_{ij} = 0$  for all  $i, j = 1, \dots, k$ . If we list the basis vectors in the order  $(e_1, \dots, e_m, f_1, \dots, f_k)$ , then  $P_U$  is a  $(m+k) \times (m+k)$  diagonal matrix with the diagonal elements in the top  $m$  rows equal to 1, and all other elements equal to zero.

In this basis, we see that  $\det P_U = 0$  (because there is at least one zero element on the diagonal) and  $\text{tr } P_U = m$ , which is the dimension of the subspace  $U$ . As the trace and determinant are invariant under a change of basis, these properties apply to any basis.

## REFERENCES

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