

PROJECTION OPERATORS

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Continuing from our examination of orthonormal bases and the orthogonal complement in a vector space V , we can now look at the orthogonal projection, sometimes known in physics as a projection operator.

Suppose we have defined a subspace U of V and its orthogonal complement U^\perp , so that $V = U \oplus U^\perp$. We can define a linear operator P_U called the *orthogonal projection* operator. It has the property that, given any vector $v \in V$, it 'projects' out the component of v that lies in U . That is, if we write

$$v = u + w \tag{1}$$

where $u \in U$ and $w \in U^\perp$, then

$$P_U v = u \tag{2}$$

In what follows, we will usually refer to P_U as just a 'projection operator', although a projection operator in physics almost always refers to an *orthogonal* projection operator.

An example of a projection operator is an operator in 3-d space that projects a vector onto the xy plane. Then the xy plane is the subspace U and the z axis is the orthogonal complement U^\perp .

From the definition of P_U we can list a few properties:

- (1) P_U is not surjective, that is, its range is smaller than the entire space V .
- (2) P_U is not injective, since it maps all vectors $u + w$ to u , for all $w \in U^\perp$. Thus it is a many-to-one mapping.
- (3) P_U is not invertible, since it is not injective. Another way of looking at this is to consider a projection in the real world where we use a flashlight to project the image of some 3-dimensional object onto a plane, creating a shadow. We cannot tell from the shadow alone what type of 3-dim object was projected. For example, if the shadow is a circle, the 3-dim object could be a sphere, cylinder, cone or any other object that could cast a circular shadow if illuminated from the right angle.

- (4) Its null space is $\text{null } P_U = U^\perp$.
- (5) Once P_U is applied to any vector v , all subsequent applications of P_U have no effect. That is, once you've projected out the component of v that lies in U , all further projections into U just give the same result. In other words $P_U^n = P_U$ for all integers $n > 0$.
- (6) $|P_U v| \leq |v|$. That is, the 'length' of a projection must be less than or equal to the length of the original vector. This follows from the Pythagorean theorem, since u and w are orthogonal, so $|v|^2 = |u|^2 + |w|^2 \geq |u|^2 = |P_U v|^2$. Geometrically, a projection operator cannot increase the 'length' (norm) of a vector. This property relies on the fact that the projection is an orthogonal projection. Other projections can increase the length of a vector (think of the shadow cast by a stick; if the surface onto which the shadow falls is nearly parallel to the direction of the incoming light, the shadow is much longer than the stick).

An explicit form for $P_U v$ can be obtained from the decomposition we had earlier

$$v = \underbrace{\sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U} + \underbrace{v - \sum_{i=1}^n \langle e_i, v \rangle e_i}_{\in U^\perp} \quad (3)$$

From this,

$$P_U v = \sum_{i=1}^n \langle e_i, v \rangle e_i \quad (4)$$

From the definition, it seems reasonable that a vector space V can be decomposed into a direct sum of range P_U and null P_U . We can in fact prove this.

Theorem 1. *P is an orthogonal projection within the vector space V if*

$$V = \text{null } P \oplus \text{range } P \quad (5)$$

Proof. We can take the subspace $U = \text{range } P$. From our earlier theorem, we know that $V = U \oplus U^\perp$, so we need to show that $U^\perp = \text{null } P$. Since $Pw = 0$ for any $w \in U^\perp$, then $\text{null } P \subset U^\perp$, but are there vectors in U^\perp that are not in $\text{null } P$? Suppose there is such a vector $x \in U^\perp$ such that $Px \neq 0$. For such a vector, we can decompose it into $x = x' + x''$ where $x' \in \text{null } P$ and $x'' \in \text{range } P$, with $x'' \neq 0$ (since if $x'' = 0$, then x would be in $\text{null } P$, contrary to our assumption).

As $x \in U^\perp$, $\langle x, u \rangle = 0$ for all $u \in U = \text{range } P$. Therefore $\langle x, u \rangle = \langle x' + x'', u \rangle = \langle x', u \rangle + \langle x'', u \rangle = 0$. Since $x' \in \text{null } P$, $\langle x', u \rangle = 0$ (as $x' \in$

U^\perp). Therefore we must have $\langle x'', u \rangle = 0$, implying that $x'' \in U^\perp$ also. Thus $x'' \in U$ and $x'' \in U^\perp$, but the only vector that can be in both a subspace and its orthogonal complement is 0, so $x'' = 0$, which contradicts our assumption above. \square

From property 5 above, we must have $P_U^2 = P_U$, which implies that the eigenvalues of P_U are 0 and 1. The eigenvectors belong to either the subspace U (for eigenvalue 1) or to the orthogonal complement U^\perp (for eigenvalue 0).

The orthonormal basis of a vector space V can be divided into two separate lists of vectors, with one list (e_1, \dots, e_m) spanning the subspace U and the other list (f_1, \dots, f_k) spanning U^\perp . A matrix representation of P_U can be obtained by considering the action of P_U on each of the basis vectors from the two subspaces. We have

$$P_U e_i = e_i \quad (6)$$

$$P_U f_i = 0 \quad (7)$$

In general, the matrix representation of an operator T is defined in terms of its action on the basis vectors v_i by

$$v'_j = \sum_{i=1}^n T_{ij} v_i \quad (8)$$

For a projection operator, we can see that this means that for the m basis vectors (e_1, \dots, e_m) we must have $P_{ij} = \delta_{ij}$ for all $i, j = 1, \dots, m$, while for the k basis vectors (f_1, \dots, f_k) we must have $P_{ij} = 0$ for all $i, j = 1, \dots, k$. If we list the basis vectors in the order $(e_1, \dots, e_m, f_1, \dots, f_k)$, then P_U is a $(m+k) \times (m+k)$ diagonal matrix with the diagonal elements in the top m rows equal to 1, and all other elements equal to zero.

In this basis, we see that $\det P_U = 0$ (because there is at least one zero element on the diagonal) and $\text{tr } P_U = m$, which is the dimension of the subspace U . As the trace and determinant are invariant under a change of basis, these properties apply to any basis.

REFERENCES

- (1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 6.
- (2) Zwiebach, Barton, Online course *Mastering Quantum Mechanics Part 1: Wave Mechanics*. Archive available here.
- (3) Landi, Giovanni & Zampini, Alessandro (2018), *Linear Algebra and Analytic Geometry for Physical Sciences*, Chapter 10.

- (4) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.