RATIO TEST FOR SERIES

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The *ratio test* is a theorem that can be used to determine if an infinite series is convergent or divergent.

Theorem 1. *(Ratio test) Suppose the terms c_j of a series $S=\sum_{j=0}^{\infty}c_j$ have the property that*

$$\lim_{j\to\infty}\left|\frac{c_{j+1}}{c_j}\right|=L \tag{1}$$

Then the series converges if $L<1$ and diverges if $L>1$.

Proof. If $L<1$, then we can find a number $\varepsilon>0$ such that $L+\varepsilon<1$. Suppose that

$$\left|\frac{c_{j+1}}{c_j}\right|<L+\varepsilon<1 \tag{2}$$

for all $j>J$ where J is some fixed integer. Then

$$\begin{aligned}
|c_{J+1}| &< |c_J|(L+\varepsilon)\\
|c_{J+2}| &< |c_{J+1}|(L+\varepsilon)<|c_J|(L+\varepsilon)^2\\
&\vdots\\
|c_{J+k}| &< |c_J|(L+\varepsilon)^k
\end{aligned} \tag{3}$$

For the series, this means that

$$\sum_{j=0}^{\infty}|c_j| = \sum_{j=0}^{J-1}|c_j|+\sum_{k=J}^{\infty}|c_k| \tag{4}$$

$$< \sum_{j=0}^{J-1}|c_j|+|c_J|\sum_{k=J}^{\infty}(L+\varepsilon)^{k-J} \tag{5}$$

The first sum on the RHS is finite so always converges. The second sum is a truncated geometric series which converges if each term is of the form

$|a_k| < 1$, which is the case here. Therefore the series $\sum_{j=0}^{\infty} c_j$ converges if 1 is true. □

Here are some examples of using the ratio test to prove convergence.

Example 1. We have

$$\sum_{j=0}^{\infty} \frac{1}{j!} \tag{6}$$

Readers may recognize this as the series expansion of e, the base of natural logarithms, so we know the series converges. However, the ratio test gives us

$$\left|\frac{c_{j+1}}{c_j}\right| = \frac{1}{j+1} < 1 \tag{7}$$

for $j \geq 1$.

Example 2. We have

$$\sum_{k=0}^{\infty} \frac{(3+i)^k}{k!} \tag{8}$$

Again, this is the series form of $e^{3+i} = e^3 (\cos 1 + i \sin 1)$. The ratio test gives

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{3+i}{k+1}\right| = \frac{\sqrt{10}}{k+1} \tag{9}$$

which goes to zero for large k.

Example 3. We have

$$\sum_{j=0}^{\infty} \frac{j^2}{4^j} \tag{10}$$

The ratio test gives

$$\left|\frac{c_{j+1}}{c_j}\right| = \frac{(j+1)^2}{4j^2} \tag{11}$$

This has a limit of

$$\lim_{j\to\infty} \frac{(j+1)^2}{4j^2} = \frac{1}{4} \tag{12}$$

so the series converges. Maple gives the value of

$$\sum_{j=0}^{\infty} \frac{j^2}{4^j} = \frac{20}{27} \tag{13}$$

We can actually derive a formula for this as follows. Start with the geometric series

$$\sum_{j=0}^{\infty} r^j = \frac{1}{1-r} \tag{14}$$

Take the derivative

$$\sum_{j=1}^{\infty} j r^{j-1} = \frac{1}{(1-r)^2} \tag{15}$$

Multiply by r

$$\sum_{j=1}^{\infty} j r^j = \frac{r}{(1-r)^2} \tag{16}$$

Take the derivative again

$$\sum_{j=1}^{\infty} j^2 r^{j-1} = \frac{1+r}{(1-r)^3} \tag{17}$$

Multiply by r again

$$\sum_{j=1}^{\infty} j^2 r^j = \frac{r(1+r)}{(1-r)^3} \tag{18}$$

We can now set $r = \frac{1}{4}$ to get

$$\sum_{j=0}^{\infty} \frac{j^2}{4^j} = \frac{5/16}{27/64} = \frac{20}{27} \tag{19}$$

Example 4. We have

$$\sum_{k=1}^{\infty} \frac{k!}{k^k} \tag{20}$$

The ratio test gives

$$\left|\frac{c_{k+1}}{c_k}\right| = (k+1)\frac{k^k}{(k+1)^{k+1}} = \frac{k^k}{(k+1)^k} \tag{21}$$

Dividing top and bottom by k^k we get

$$\lim_{k\to\infty}\frac{k^k}{(k+1)^k}=\lim_{k\to\infty}\frac{1}{(1+1/k)^k} \tag{22}$$

The limit

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k \tag{23}$$

is one of the definitions of e, the base of natural logarithms. Thus the limit 22 is $1/e<1$, so the series converges. I don't know what the actual value of the series is; Maple is unable to calculate it in closed form anyway. The sum of the first 100 terms gives a value of 1.879853862, which matches the sum of the first 500 terms to the same number of decimal places. I don't recognize this as an obvious quantity.

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