RECENTERING POLYNOMIALS

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Post date: 13 February 2025.

We can generalize the Taylor form of polynomials as follows. Suppose we have a polynomial of form

$$p(z)=a_0+a_1z+\ldots+a_nz^n \tag{1}$$

This polynomial is centred at $z_0=0$. Now we want to recentre it at $z_0=1$, so it has the form

$$p_n(z)=c_0+c_1(z-1)+\ldots+c_n(z-1)^n \tag{2}$$

The easiest way to do this is by taking the Taylor series of 1 about $z_0=1$. We have

$$p(z)=\sum_{j=0}^{n}\frac{p^{(j)}(1)}{j!}(z-1)^j \tag{3}$$

The derivatives are, from 1

$$\begin{aligned}
p(1)&=\sum_{\ell=0}^{n}a_\ell\\
p'(1)&=\sum_{\ell=1}^{n}\ell a_\ell\\
p''(1)&=\sum_{\ell=2}^{n}\ell(\ell-1)a_\ell\\
&\vdots\\
p^{(k)}(1)&=\sum_{\ell=k}^{n}\ell(\ell-1)\ldots(\ell-k+1)a_\ell
\end{aligned} \tag{4}$$

From 2 we have

$$\begin{aligned} c_0 &= p(1) \\ c_1 &= p'(1) \\ c_2 &= \frac{p''(1)}{2!} \\ &\vdots \\ c_k &= \frac{p^{(k)}(1)}{k!} \end{aligned} \tag{5}$$

So the relation between the two sets of coefficients is, for $k > 0$

$$c_k = \frac{1}{k!}\sum_{\ell=k}^{n} \ell(\ell-1)\ldots(\ell-k+1)\,a_\ell \tag{6}$$

$$= \sum_{\ell=k}^{n} \binom{\ell}{k} a_\ell \tag{7}$$

Example 1. Recentre about $z_0 = 1$ the polynomial

$$p(z) = 3 - 4z + 5z^2 - 12z^3 \tag{8}$$

We get

$$\begin{aligned} p(1) &= -8 \\ p'(1) &= \sum_{\ell=1}^{3} \ell a_\ell = -4 + 5\times 2 + 3\times(-12) = -30 \\ p''(1) &= \sum_{\ell=2}^{3} \ell(\ell-1)\,a_\ell = 2\times 1\times 5 + 3\times 2\times(-12) = -62 \\ p^{(3)}(1) &= \sum_{\ell=3}^{3} \ell(\ell-1)(\ell-2)\,a_\ell = 3\times 2\times 1\times(-12) = -72 \end{aligned} \tag{9}$$

Therefore

$$p(z) = -8 - 30(z-1) - \frac{62}{2!}(z-1)^2 - \frac{72}{3!}(z-1)^3 \tag{10}$$

$$= -8 - 30(z-1) - 31(z-1)^2 - 12(z-1)^3 \tag{11}$$

To recentre a polynomial about some other point z_0 we can find its Taylor series relative to that point. That is

$$p(z) = \sum_{\ell=0}^{n} \frac{p^{(\ell)}(z_0)}{\ell!}(z - z_0)^{\ell} \tag{12}$$

So we need to calculate the derivative terms and evaluate them at $z = z_0$. The formula relating the two sets of coefficients would be more complicated than 7 for $z_0 = 1$, but the procedure is the same.

Example 2. Recentre about $z_0 = -2$ the polynomial 8. If we grind through the derivatives (I used Maple, but the derivatives are the standard ones for polynomials):

$$\begin{aligned} p(-2) &= 127 \\ p'(-2) &= -168 \\ p''(-2) &= 154 \\ p^{(3)}(-2) &= -72 \end{aligned} \tag{13}$$

So we have

$$p(z) = 127 - 168(z+2) + \frac{154}{2!}(z+2)^2 - \frac{72}{3!}(z+2)^2 \tag{14}$$

$$= 127 - 168(z+2) + 77(z+2)^2 - 12(z+2)^2 \tag{15}$$