RECIPROCAL EXPONENTS

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The function $e^{1/z}$ has rather bizarre behaviour near the point $z=0$. In fact, it's possible to make $e^{1/z}$ equal almost any value, even for extremely small values of z. We give a few examples where in all cases we find a z such that $|z|<0.001$.

Example 1. $e^{1/z}=i$. We know that $e^{i\pi/2}=i$, but we can add any multiple of 2π to the exponent with the same result. We can take, for example

$$z=\left(\frac{i\pi}{2}+1000\pi i\right)^{-1} \tag{1}$$

This has a magnitude

$$|z|=\frac{2}{2001\pi}=0.0003181508107... \tag{2}$$

Example 2. $e^{1/z}=-1$. We have $e^{i\pi}=-1$, but again we can add a multiple of 2π to the exponent. For example

$$z=\frac{1}{1001\pi i} \tag{3}$$

also gives $e^{1/z}=-1$. With this value of z, we have

$$|z|=\frac{1}{1001\pi}=0.0003179918942... \tag{4}$$

We can also generate very large or very small numbers for values of z such that $|z|<0.001$.

Example 3. $e^{1/z}=6.02\times10^{23}$, Avogadro's number from chemistry. Using natural logarithms, we find that

$$\ln e^{1/z}=\frac{1}{z}=\ln\left(6.02\times10^{23}\right)=54.75454440 \tag{5}$$

This gives a value of $z=0.01826332428$. To generate a z with a smaller magnitude, we can add a multiple of $2\pi i$ to the exponent. For example

$$\frac{1}{z} = 54.75454439 + 1000\pi i \tag{6}$$

gives us, to 4 significant figures:

$$z = 5.546 \times 10^{-6} - 0.0003183i \tag{7}$$

This has a magnitude of

$$|z| = 0.0003182615512... \tag{8}$$

It's worth noting here that using a package such as Maple to calculate $e^{1/z}$ for such values is very sensitive to round-off error. Maple is able to get the correct answer if we give it the value 6:

$$e^{54.75454439+1000\pi i} = 6.019999951 \times 10^{23} \tag{9}$$

but if we give it $e^{1/z}$ with z given by 7 we get

$$e^{1/z} = 3.826065167 \times 10^{23} - 4.407676611 \times 10^{23} i \tag{10}$$

The round-off error is due mostly to truncating the value of the imaginary part in 7. If we use

$$z = 5.546 \times 10^{-6} - 0.0003182i \tag{11}$$

(that is, changing the last digit of the imaginary part by only 1), we get

$$e^{1/z} = 5.989482386 \times 10^{23} + 7.861827873 \times 10^{22} i \tag{12}$$

The real part is closer to Avogadro's number, but the imaginary part is still wildly wrong.

Example 4. At the other extreme, we try $e^{1/z} = 1.6 \times 10^{-19}$, which is the proton's charge in Coulombs. Again using logarithms, we find

$$\ln e^{1/z} = \frac{1}{z} = \ln\left(1.6 \times 10^{-19}\right) = -43.27911314... \tag{13}$$

This gives a starting value of

$$z = -0.02310583391... \tag{14}$$

We can add a multiple of $2\pi i$ to $1/z$ to get a smaller value for z. For example

$$\frac{1}{z} = -43.27911314 + 1000\pi i \tag{15}$$

which gives, to 10 significant figures

$$z = -4.384258913 \times 10^{-6} - 0.0003182494879i \tag{16}$$

with

$$|z| = 0.00031827968 \quad (17)$$

In this case, the round-off error with Maple isn't quite as bad. Evaluating $e^{1/z}$ with 16 gives us

$$e^{1/z} = 1.599999996 \times 10^{-19} - 9.436691793 \times 10^{-26} i \quad (18)$$

Using a precise imaginary part gives

$$e^{-43.27911314+1000\pi i} = 1.599999996 \times 10^{-19} \quad (19)$$