REFLECTION OF COMPLEX NUMBER THROUGH A LINE

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Reflecting a complex number z across the real axis in the complex plane turns z into its complex conjugate $\overline{z}$. It's also possible to reflect z through an arbitrary line L with equation $ax+by=c$, where all quantities are real numbers.

To do this we can resort to geometry in the xy plane. Suppose that $z=x_0+iy_0$ so that it is represented by the point (x_0,y_0) in the plane. To reflect z through the line, we need to find the line P that is perpendicular to L and contains the point (x_0,y_0). We can rewrite L as

$$y=-\frac{a}{b}x+\frac{c}{b} \tag{1}$$

Thus L has slope $-\frac{a}{b}$, and P has the slope that is the negative reciprocal to that of L. Using the point-slope form of P's equation, we therefore have

$$y-y_0=\frac{b}{a}(x-x_0) \tag{2}$$

We can find the intersection of these two lines by solving 1 and 2 as simultaneous equations. This is a bit messy so I used Maple to do this, with the result that the intersection point (x_1,y_1) is given by

$$x_1=-\frac{aby_0-b^2x_0-ac}{a^2+b^2} \tag{3}$$

$$y_1=\frac{a^2y_0-abx_0+bc}{a^2+b^2} \tag{4}$$

The reflection of z through L results in travelling from (x_0,y_0) to (x_1,y_1) and then by an equal distance to the other side of (x_1,y_1). That is, the reflection point (x_2,y_2) is found from

$$x_2=x_0+2(x_1-x_0)=2x_1-x_0 \tag{5}$$

$$y_2=y_0+2(y_1-y_0)=2y_1-y_0 \tag{6}$$

Again, the algebra is somewhat messy so I used Maple to simplify things with the result

$$
\begin{aligned}
x_2 &= \frac{-a^2 x_0 + (-2by_0 + 2c)\,a + b^2 x_0}{a^2 + b^2} \\
y_2 &= \frac{-b^2 y_0 + (-2ax_0 + 2c)\,b + a^2 y_0}{a^2 + b^2}
\end{aligned}
\tag{7}
$$

Thus the reflection of z through L results in

$$z_R = x_2 + iy_2 \tag{8}$$

The original problem in Saff and Snider's book asked us to show that the reflected number is

$$z_R = \frac{2ic + (b - ai)\,\overline{z}}{b + ai} \tag{9}$$

with $\overline{z} = x_0 - iy_0$.

I'm not sure how we could derive this result, but using Maple, we can check that it matches 7 by finding the real and imaginary parts of z_R as given by 9. Doing this and simplifying the result does indeed give a match:

$$
\begin{aligned}
\Re z_R &= \frac{-a^2 x_0 + (-2by_0 + 2c)\,a + b^2 x_0}{a^2 + b^2} = x_2 \\
\Im z_R &= \frac{-b^2 y_0 + (-2ax_0 + 2c)\,b + a^2 y_0}{a^2 + b^2} = y_2
\end{aligned}
\tag{10}
$$