REMAINDER IN MACLAURIN SERIES

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In the course of proving the theorem about the convergence of Taylor series, Saff and Snider use the Cauchy integral formula

$$f(z)=\frac{1}{2\pi i}\int_{C}\frac{f(\zeta)}{\zeta-z}d\zeta \tag{1}$$

Here, the contour C is the circle with centre z_0 of radius $|\zeta-z_0|$. The point z lies within the circle, so

$$|z-z_0|<|\zeta-z_0| \tag{2}$$

The $1/(\zeta-z)$ factor in 1 can be expanded in a geometric series:

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0-(z-z_0)} \tag{3}$$

$$=\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\cdot\frac{1}{\zeta-z_0} \tag{4}$$

$$=\left[1+\frac{z-z_0}{\zeta-z_0}+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+\left(\frac{z-z_0}{\zeta-z_0}\right)^n+T_{n+1}\right]\cdot\frac{1}{\zeta-z_0} \tag{5}$$

where T_{n+1} is a remainder term, which can be calculated from the geometric series. For a general geometric series

$$S_n=\sum_{j=0}^{n}a^j=\frac{1-a^{n+1}}{1-a} \tag{6}$$

For $|a|<1$, the series can be extended to infinity, with the result

$$S=\sum_{j=0}^{\infty}a^j=\frac{1}{1-a} \tag{7}$$

The remainder is then

$$S-S_n=\frac{a^{n+1}}{1-a} \tag{8}$$

Returning to 5, we have

$$a = \frac{z - z_0}{\zeta - z_0} \tag{9}$$

$$T_{n+1} = \left[\frac{z - z_0}{\zeta - z_0}\right]^{n+1} \left[1 - \frac{z - z_0}{\zeta - z_0}\right]^{-1} \tag{10}$$

The remainder term in 1 is therefore

$$f_{Rem} = \frac{1}{2\pi i} \int_C f(\zeta) \frac{T_{n+1}}{\zeta - z_0} d\zeta \tag{11}$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{z - z_0}{\zeta - z_0}\right]^{n+1} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{12}$$

A *Maclaurin series* is just a Taylor series about the origin, that is, with $z_0 = 0$. In this case the remainder term becomes

$$T_{n+1} = \frac{(z/\zeta)^{n+1}}{1 - z/\zeta} \cdot \zeta \tag{13}$$

$$= \frac{z^{n+1}}{\zeta^{n+1}(\zeta - z)} \tag{14}$$

Thus the remainder in the Maclaurin series is 12 with $z_0 = 0$:

Hermite formula

$$f(z) - \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} = \frac{1}{2\pi i} \int_C \frac{z^{n+1}}{\zeta^{n+1}} \frac{f(\zeta)}{(\zeta - z)} d\zeta \tag{15}$$

This is known as the *Hermite formula*. The integral requires knowing $f(\zeta)$ only on the circle C, and not in any point interior to the circle. Thus it can be used to find upper bounds on the error in a truncated Maclaurin series.