RESIDUES REVISITED

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We first met residues when discussing rational functions, that is, ratios of two polynomials. It turns out that the idea of a residue is more general. The definition is

Definition 1. If a function $f(z)$ has an isolated singularity at a point z_0, then the coefficient a_{-1} in the Laurent expansion around $z=z_0$ is called the residue of f at z_0 and is denoted by $\text{Res}(f;z_0)$ or sometimes in abbreviated form as $\text{Res}(z_0)$.

A theorem (proved in Saff and Snider, section 6.1) for finding the residue at a pole is:

Theorem 1. *If f has a pole of order m at z_0 then*

$$Res(f;z_0)=\lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] \tag{1}$$

A special case of this is for a function that is the ratio of two functions $P(z)$ and $Q(z)$ (not necessarily a rational function of two polynomials) with a simple pole (order 1). Then, since $Q(z_0)=0$ (that is, the denominator must have a zero at z_0 in order for the function to have a pole there):

$$\text{Res}(f;z_0)=\lim_{z\to z_0}\left[(z-z_0)\frac{P(z)}{Q(z)}\right] \tag{2}$$

$$=\lim_{z\to z_0}\left[(z-z_0)\frac{P(z)}{Q(z)-Q(z_0)}\right] \tag{3}$$

$$=\lim_{z\to z_0}\frac{P(z)}{\frac{Q(z)-Q(z_0)}{z-z_0}} \tag{4}$$

$$=\frac{P(z_0)}{Q'(z_0)} \tag{5}$$

Here are a few examples.

Example 1. Given

$$f(z)=\frac{e^{3z}}{z-2}\equiv\frac{P(z)}{Q(z)} \tag{6}$$

This has a simple pole at $z_0=2$ so we have

$$\text{Res}(2)=\frac{e^{3\times 2}}{1}=e^6 \tag{7}$$

Example 2. Given

$$f(z)=\frac{z+1}{z^2-3z+2}\equiv\frac{P(z)}{Q(z)} \tag{8}$$

The denominator factors into

$$z^2-3z+2=(z-1)(z-2) \tag{9}$$

so we have simple poles at $z_0=1$ and $z_0=2$. We have

$$Q'(z)=2z-3 \tag{10}$$

so

$$\text{Res}(1)=\frac{1+1}{2-3}=-2 \tag{11}$$

$$\text{Res}(2)=\frac{2+1}{4-3}=3 \tag{12}$$

Example 3. Given

$$f(z)=\frac{\cos z}{z^2} \tag{13}$$

This has a pole of order 2 at $z=0$. It's easiest to expand $\cos z$ in its series.

$$\frac{\cos z}{z^2}=\frac{1}{z^2}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right) \tag{14}$$

$$=\frac{1}{z^2}-\frac{1}{2}+\frac{z^2}{4!}-\ldots \tag{15}$$

Thus the coefficient of z^{-1} is 0, so $\text{Res}(0)=0$.

Example 4. Given

$$f(z)=\left(\frac{z-1}{z+1}\right)^{3} \tag{16}$$

This has a pole of order 3 at $z=-1$. We can find the residue from 1. We have

$$\operatorname{Res}(-1)=\lim_{z\to-1}\frac{1}{2!}\frac{d^{2}}{dz^{2}}\left[(z+1)^{3}\left(\frac{z-1}{z+1}\right)^{3}\right] \tag{17}$$

$$=\lim_{z\to-1}\frac{1}{2!}(6z-6) \tag{18}$$

$$=-6 \tag{19}$$

Example 5. Given

$$f(z)=\frac{e^{z}}{z(z+1)^{3}}\equiv\frac{P(z)}{Q(z)} \tag{20}$$

This has a simple pole at $z=0$ and a pole of order 3 at $z=-1$. For the simple pole, we use 5.

$$Q'(z)=(z+1)^{3}+3z(z+1)^{2} \tag{21}$$

$$Q'(0)=1 \tag{22}$$

$$P(0)=1 \tag{23}$$

so

$$\operatorname{Res}(0)=1 \tag{24}$$

For the pole at $z=-1$ we use 1:

$$\operatorname{Res}(-1)=\lim_{z\to-1}\frac{1}{2!}\frac{d^{2}}{dz^{2}}\left[(z+1)^{3}\frac{e^{z}}{z(z+1)^{3}}\right] \tag{25}$$

$$=\lim_{z\to-1}\left[\frac{e^{z}}{2z}-\frac{e^{z}}{z^{2}}+\frac{e^{z}}{z^{3}}\right] \tag{26}$$

$$=-\frac{5}{2e} \tag{27}$$

Example 6. Given

$$f(z)=\sin\left(\frac{1}{3z}\right) \tag{28}$$

This has an essential singularity at $z=0$. We can expand it in a series, but we need only the term in z^{-1} to get the residue.

$$\sin\left(\frac{1}{3z}\right)=\frac{1}{3z}-\frac{1}{3!\,(3z)^3}+\ldots \tag{29}$$

The residue is therefore

$$\text{Res}(0)=\frac{1}{3} \tag{30}$$

Example 7. Given

$$f(z)=\tan z \tag{31}$$

We write this as

$$\tan z=\frac{\sin z}{\cos z}\equiv\frac{P(z)}{Q(z)} \tag{32}$$

and use 5 since $\cos z$ has simple zeroes at $z_0=\left(n+\frac{1}{2}\right)\pi$. We have

$$\text{Res}\left(\left(n+\frac{1}{2}\right)\pi\right)=\lim_{z\to\left(n+\frac{1}{2}\right)\pi}\frac{\sin z}{-\sin z} \tag{33}$$

$$=-1 \tag{34}$$

Example 8. Given

$$f(z)=\frac{z-1}{\sin z}\equiv\frac{P(z)}{Q(z)} \tag{35}$$

As $\sin z$ has simple zeroes at $z_0=n\pi$ we use 5 to get

$$\text{Res}(n\pi)=\lim_{z\to n\pi}\frac{z-1}{\cos(z)} \tag{36}$$

$$=(-1)^n(n\pi-1) \tag{37}$$

Example 9. Given

$$f(z)=\frac{z^2}{1-\sqrt{z}} \tag{38}$$

We can rationalize the denominator as follows:

$$\frac{z^2}{1-\sqrt{z}}=\frac{z^2}{1-\sqrt{z}}\times\frac{1+\sqrt{z}}{1+\sqrt{z}} \tag{39}$$

$$=\frac{z^2\left(1+\sqrt{z}\right)}{1-z} \tag{40}$$

$$=-\frac{z^2\left(1+\sqrt{z}\right)}{z-1}\equiv-\frac{P(z)}{Q(z)} \tag{41}$$

This has a simple pole at $z=1$ so we use 5 to get

$$Q'(z)=1 \tag{42}$$

$$P(1)=2 \tag{43}$$

so

$$\text{Res}(1)=-2 \tag{44}$$

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