RESIDUES

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Suppose we have a rational function $R_{m,n}(z) = P(z)/Q(z)$ with the degree m of P less than the degree n of Q. If ζ_k is a pole of $R_{m,n}$, then the coefficient of $1/(z-\zeta_k)$ in the partial fraction decomposition of $R_{m,n}$ is called the *residue* of ζ_k and is denoted $\text{Res}(\zeta_k)$. Note that it's only the term in the partial fraction decomposition that has a linear term in the denominator that has a residue. A pole with multiplicity greater than 1 will typically have more than one term in the partial fraction decomposition, but for the purposes of finding the residue, only the linear term need be calculated. The formula for calculating the coefficient in a partial fraction decomposition is

$$A_s^{(j)} = \lim_{z\to\zeta_j} \frac{1}{s!}\frac{d^s}{dz^s}\left[(z-\zeta_j)^{d_j} R_{m,n}(z)\right] \tag{1}$$

If a pole has multiplicity d_j, then to find the coefficient of the linear denominator, we use 1 with $s = d_j - 1$.

Here are a few examples. I've used Maple to do the derivatives and simplify the calculations in each case.

Example 1. $\text{Res}(i)$ for $R_{1,3}(z) = \frac{2z+3}{(z-i)(z^2+1)}$. We first factor the denominator to get

$$R_{1,3}(z) = \frac{2z+3}{(z-i)^2(z+i)} \tag{2}$$

The pole at $z = i$ has multiplicity 2, so the residue is

$$\text{Res}(i) = \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2 R_{1,3}(z)\right] \tag{3}$$

$$= \lim_{z\to i}\left[\frac{2}{z+1} - \frac{2z+3}{(z+1)^2}\right] \tag{4}$$

$$= \frac{i}{2} \tag{5}$$

Example 2. $\text{Res}(-1)$ for $R_{3,6}(z) = \frac{z^3+4z+9}{(2z+2)(z-3)^5}$. In this case we can write

$$R_{3,6}(z)=\frac{z^3+4z+9}{2(z+1)(z-3)^5} \tag{6}$$

so the pole at $z=-1$ has multiplicity 1, and we have

$$\begin{aligned}\operatorname{Res}(-1) &= \lim_{z\to-1}\left[(z+1)R_{3,6}(z)\right] &(7)\\ &= \lim_{z\to-1}\frac{z^3+4z+9}{2(z-3)^5} &(8)\\ &= -\frac{1}{512} &(9)\end{aligned}$$

For reference, we could also calculate the residue at $z=3$ by the formula

$$\begin{aligned}\operatorname{Res}(3) &= \frac{1}{4!}\lim_{z\to3}\frac{d^4}{dz^4}\left[(z-3)^5R_{3,6}(z)\right] &(10)\\ &= \frac{1}{512} &(11)\end{aligned}$$

I used Maple to do the calculations as the fourth derivative gets quite messy.

Example 3. $\operatorname{Res}(0)$ for $R_{2,4}(z)=\frac{2z^2+3}{z^2\left(z^2+2z+i\right)}$. In this case, the pole at $z=0$ has multiplicity 2 so we have

$$\begin{aligned}\operatorname{Res}(0) &= \lim_{z\to0}\frac{d}{dz}\left[z^2R_{2,4}(z)\right] &(12)\\ &= \lim_{z\to0}\left[\frac{4z}{z^2+2z+i}-\frac{\left(2z^2+3\right)(2z+2)}{\left(z^2+2z+i\right)^2}\right] &(13)\\ &= 6 &(14)\end{aligned}$$

Example 4. $\operatorname{Res}(3i)$ for $R_{2,4}(z)=\frac{z^2-9}{\left(z^2+9\right)^2}$. Factoring the denominator we have

$$R_{2,4}(z)=\frac{z^2-9}{(z+3i)^2(z-3i)^2} \tag{15}$$

The residue is then

$$\text{Res}(3i) = \lim_{z\to 3i}\frac{d}{dz}\left[(z-3i)^2 R_{2,4}(z)\right] \tag{16}$$

$$= \lim_{z\to 3i}\left[\frac{2z}{(z+3i)^2} - \frac{2\left(z^2-9\right)}{(z+3i)^3}\right] \tag{17}$$

$$= 0 \tag{18}$$

Thus in this case, there is no term in the partial fraction decomposition with a linear term in the denominator. For reference, the partial fraction decomposition here is

$$R_{2,4}(z) = \frac{1}{2(z+3i)^2} + \frac{1}{2(z-3i)^2} \tag{19}$$

Example 5. $\text{Res}(0)$ for $R_{3,4}(z) = \frac{2z^3+3}{z^3(z+1)}$. We get

$$\text{Res}(0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left[z^3 R_{3,4}(z)\right] \tag{20}$$

$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{12z}{z+1} - \frac{12z^2}{(z+1)^2} + \frac{2\left(2z^3+3\right)}{(z+1)^3}\right] \tag{21}$$

$$= 3 \tag{22}$$

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