ROOTS OF COMPLEX NUMBERS

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From De Moivre's formula and the exponential form of a complex number z, we can calculate powers in the form

$$z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta) \tag{1}$$

Calculating roots of complex numbers can be done the same way, although we need to be aware that most complex numbers have more than one root. The simplest example is the square root of a positive real number a, which has the two values $\pm\sqrt{a}$.

Consider the nth roots of 1. Using exponential notation, we can write 1 as

$$1 = e^{2k\pi i} \tag{2}$$

where k is an integer. The nth roots are then

$$1^{1/n} = e^{2k\pi i/n} \tag{3}$$

Here, k can take on the values $0, 1, \ldots, n-1$ to give n distinct values for $1^{1/n}$. For higher values of k the cycle of roots repeats, so no new values are found.

Example 1. Find the five values of $1^{1/5}$. We have

$$1^{1/5} = e^{2k\pi i/5} \tag{4}$$

$$= 1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5} \tag{5}$$

These values are equivalent to

$$1, \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} \tag{6}$$

Example 2. We can use similar methods to find the roots of any complex number. For $z = -16$, we have

$$-16 = 16e^{\pi i} \tag{7}$$

so

$$(-16)^{1/4} = 2e^{(\pi+2k\pi)i/4} \tag{8}$$

$$= 2e^{\pi i/4}, 2e^{3\pi i/4}, 2e^{5\pi i/4}, 2e^{7\pi i/4} \tag{9}$$

Example 3. A more involved example is $\left(\frac{2i}{1+i}\right)^{1/6}$. First, convert the quotient into exponential form. We have

$$\frac{2i}{1+i} = \frac{2e^{\pi i/2}}{\sqrt{2}e^{\pi i/4}} \tag{10}$$

$$= \sqrt{2}e^{\pi i/4} \tag{11}$$

Therefore

$$\left(\frac{2i}{1+i}\right)^{1/6} = 2^{1/12}e^{(2k\pi+\pi/4)i/6} \tag{12}$$

$$= 2^{1/12}\left[e^{\pi i/24}, e^{9\pi i/24}, e^{17\pi i/24}, e^{25\pi i/24}, e^{33\pi i/24}, e^{41\pi i/24}\right] \tag{13}$$

We now define the root of 1 with $k = 1$ in 3 as

$$\omega_n \equiv e^{2\pi i/n} \tag{14}$$

An interesting result is that

$$\sum_{k=0}^{n-1} \omega_n^k = 0 \tag{15}$$

We can see this by using the formula for a geometric series:

$$\sum_{k=0}^{n-1} \omega_n^k = \frac{\omega_n^n - 1}{\omega_n - 1} \tag{16}$$

Since ω_n is an nth root of 1, $\omega_n^n = 1$, so $\omega_n^n - 1 = 0$. Since ω_n is the lowest root of 1 that is not 1 itself, $\omega_n \neq 1$ so we have

$$\sum_{k=0}^{n-1} \omega_n^k = \frac{\omega_n^n - 1}{\omega_n - 1} = 0 \tag{17}$$

This has the geometric interpretation that the nth roots of 1 form the vertices of an n-sided regular polygon with centre at the origin. Thus the sum of all the vectors pointing to these vertices is equivalent (in physics terms) to the centre of mass of n masses placed at the vertices, which is the centre of the polygon.

Example 4. We can verify 17 for the cases $n = 3$ and $n = 4$. For $n = 3$

$$\omega_3 = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \tag{18}$$

So we have

$$1 + \omega_3 + \omega_3^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) = 0 \tag{19}$$

For $n = 4$

$$\omega_4 = e^{2\pi i/4} = i \tag{20}$$

and we have

$$1 + \omega_4 + \omega_4^2 + \omega_4^3 = 1 + i - 1 - i = 0 \tag{21}$$