ROUCHÉ'S THEOREM

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Theorem 1. *(Rouché's theorem) Suppose that $f(z)$ and $h(z)$ are functions that are analytic within and on a simple closed contour C (such as a circle). Suppose further that at all points on the contour C (but not necessarily inside C) that*

$$|h(z)| < |f(z)| \tag{1}$$

Then f and $f+h$ have the same total number of zeroes (counting multiplicities) inside C.

Note that 1 applies only on the contour and not necessarily inside it, and that the inequality is a strict inequality (not 'less than or equal'). Note also that both f and h must be analytic, so they don't have any poles or other singularities.

The application of Rouché's theorem usually involves a function f and a perturbation function h, for one of which we can determine the actual number of zeroes within a contour.

Example 1. For the function

$$g(z) = z^6 + 4z^2 - 1 \tag{2}$$

find the number of zeroes in the disk $|z| < 1$.

The trick in problems of this sort is identifying which part is the perturbation h. Here, we take

$$\begin{aligned} f(z) &= 4z^2 - 1 \\ h(z) &= z^6 \end{aligned} \tag{3}$$

We see that $f(z)$ has two zeroes, at $z = \pm\frac{1}{2}$, both of which lie inside the contour $|z| = 1$. On the contour, we have, using the reverse triangle inequality

$$|f(z)| = \left|4z^2 - 1\right| \tag{4}$$

$$
\begin{aligned}
&\geq \left|4\left|z^2\right|-1\right| && (5)\\
&= |4\times 1-1| && (6)\\
&= 3 && (7)
\end{aligned}
$$

For the perturbation, we have, on $|z|=1$

$$|h(z)| = \left|z^6\right| = 1 \tag{8}$$

Thus $|h(z)| < |f(z)|$ on the contour, so the number of zeroes of $g(z) = f(z)+h(z)$ in the disk $|z|<1$ is the same as the number of zeroes of f in $|z|<1$, which is 2. As g is a 6th degree polynomial, it has 6 roots, so we know that 4 of these roots lie outside $|z|=1$.

Using Maple, we find that the roots are at (approximate values)

$$z = \begin{cases} \pm 0.4962 \\ -0.9726 \pm 1.0340i \\ 0.9726 \pm 1.0340i \end{cases} \tag{9}$$

The magnitudes of these values are 0.4962 for the first two, and 1.4195 for the remaining 4, so there are indeed 2 zeroes within $|z|=1$.

Example 2. For the function

$$g(z) = z^3 + 9z - 27 \tag{10}$$

show that there are no roots in the disk $|z|<2$.

We take

$$
\begin{aligned}
f(z) &= 9z-27\\
h(z) &= z^3
\end{aligned} \tag{11}
$$

We see that $f(z)$ has one zero, at $z=3$, which lies outside the disk. We have, on the disk $|z|=2$

$$
\begin{aligned}
|f(z)| &= |9z-27| && (12)\\
&\geq |9|z|-27| && (13)\\
&= |9\times 2-27| && (14)\\
&= 9 && (15)
\end{aligned}
$$

For the perturbation on $|z|=2$

$$|h(z)| = \left|z^3\right| = 8 \tag{16}$$

Thus $|h(z)| < |f(z)|$ on the contour $|z| = 2$, so $g = f + h$ has the same number of zeroes as f, which is zero.

Using Maple, we find the three roots are at

$$z = \begin{cases} -2.04698 \\ 1.02349 \pm 3.4846i \end{cases} \tag{17}$$

with magnitudes 2.04698 for the first root and 3.6318 for the other two, so there are no roots within $|z| = 2$.

Example 3. For the function

$$g(z) = z^6 - 5z^2 + 10 \tag{18}$$

show that all the roots lie in the annulus $1 < |z| < 2$.

Since g is a 6th order polynomial, it has 6 roots. Thus we take

$$\begin{aligned} f(z) &= z^6 \\ h(z) &= -5z^2 + 10 \end{aligned} \tag{19}$$

Thus f has 6 zeroes (the zero at $z = 0$ has multiplicity 6). On the circle $|z| = 2$ we have

$$|f(z)| = 2^6 = 64 \tag{20}$$

$$|h(z)| = \left|-5z^2 + 10\right| \tag{21}$$

$$\geq \left|5\left|z^2\right| + 10\right| \tag{22}$$

$$= |5 \times 4 + 10| \tag{23}$$

$$= 30 \tag{24}$$

Thus $|h(z)| < |f(z)|$ on $|z| = 2$, so $g = f + h$ has all 6 zeroes inside $|z| = 2$.

Next, we need to show that there are no zeroes inside $|z| = 1$. To do this, we reverse the roles of f and h, so we have

$$\begin{aligned} f(z) &= -5z^2 + 10 \\ h(z) &= z^6 \end{aligned} \tag{25}$$

We see that $f(z)$ has two roots, at $z = \pm\sqrt{2}$, which lie outside $|z| = 1$, so there are no roots of f inside $|z| = 1$. Then, on $|z| = 1$, we have

$$|f(z)| = \left|-5z^2 + 10\right| \tag{26}$$

$$\geq \left|5\left|z^2\right|+10\right| \tag{27}$$

$$= |5\times 1+10| \tag{28}$$

$$= 15 \tag{29}$$

$$|h(z)| = \left|z^6\right| = 1 \tag{30}$$

Thus on $|z|=1$, $|h(z)|<|f(z)|$, so $g=f+h$ has no roots in the disk $|z|<1$. Thus all the roots of 18 lie in the annulus $1<|z|<2$.

Using Maple, the roots of 18 are

$$z = \begin{cases} 1.1859 \pm 0.53844i \\ 0.17245 \pm 1.49938i \\ -1.35833 \pm 0.862i \end{cases} \tag{31}$$

The magnitudes are 1.3024 for the first pair, 1.5093 for the second pair and 1.6088 for the third pair, verifying that all roots do indeed lie within the annulus.