SCALAR AND VECTOR PRODUCTS OF COMPLEX NUMBERS

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Post date: 10 November 2024.

A complex number can be represented as a vector in the complex plane, extending from the origin to the point $z = x + iy$. Given this interpretation, we can see how the scalar and vector products of linear algebra can be written in terms of complex numbers.

Consider first the scalar product. For two vectors $v_1 = [x_1, y_1]$ and $v_2 = [x_2, y_2]$, the scalar product is

$$v_1 \cdot v_2 = x_1 x_2 + y_1 y_2 \tag{1}$$

If we represent these vectors by complex numbers z_1 and z_2 and form the product

$$\overline{z}_1 z_2 = (x_1 - iy_1)(x_2 + iy_2) \tag{2}$$

$$= x_1 x_2 + y_1 y_2 + i(x_1 y_2 - x_2 y_1) \tag{3}$$

we see that the scalar product is the real part. That is

$$v_1 \cdot v_2 = \Re(\overline{z}_1 z_2) \tag{4}$$

Theorem 1. *Two vectors have zero scalar product if and only if they are orthogonal.*

Proof. An 'if and only if' proof requires proof in both directions.

From the polar representation of a complex number we see that z_1 is orthogonal to z_2 if its argument is $\frac{\pi}{2}$ greater or less than that of z_2. Since i has an argument of $\frac{\pi}{2}$, this means that z_1 is orthogonal to z_2 if

$$z_1 = icz_2 \tag{5}$$

where c is a real number. Plugging this into 3 we have

$$\overline{z}_1 z_2 = -ic\overline{z}_2 z_2 \tag{6}$$

$$= -ic|z_2|^2 \tag{7}$$

Thus $\Re(\overline{z}_1 z_2) = 0$ and thus if $z_1 \perp z_2$ the scalar product is zero.

To prove the opposite direction, it is easier to use polar form. That is, we have

$$\begin{aligned} z_1 &= r_1 \text{cis}\theta_1 \\ z_2 &= r_2 \text{cis}\theta_2 \end{aligned} \tag{8}$$

The product $\overline{z}_1 z_2$ is then

$$\overline{z}_1 z_2 = r_1 r_2 \text{cis}\left(\theta_2 - \theta_1\right) \tag{9}$$

If this is zero, then

$$\Re\left(\overline{z}_1 z_2\right) = r_1 r_2 \cos\left(\theta_2 - \theta_1\right) = 0 \tag{10}$$

Assuming that $r_1 \neq 0$ and $r_2 \neq 0$, we must therefore have

$$\cos\left(\theta_2 - \theta_1\right) = 0 \tag{11}$$

or

$$\theta_2 = \theta_1 + \frac{\pi}{2} + k\pi \tag{12}$$

for integer k. Thus requiring the scalar product to be zero implies that $z_1 \perp z_2$. □

The vector product of v_1 and v_2 as given above has components

$$v_1 \times v_2 = \left[0, 0, x_1 y_2 - x_2 y_1\right] \tag{13}$$

From 3 we see that

$$\left[v_1 \times v_2\right]_3 = \Im\left(\overline{z}_1 z_2\right) \tag{14}$$

Theorem 2. *The complex numbers represented by v_1 and v_2 are parallel (or antiparallel) if and only if $\Im\left(\overline{z}_1 z_2\right) = 0$.*

Proof. The number z_1 is parallel to z_2 if $z_1 = c z_2$ for some real number c. In this case

$$\overline{z}_1 z_2 = c \overline{z}_2 z_2 \tag{15}$$

$$= c\left|z_2\right|^2 \tag{16}$$

Thus $\Im\left(\overline{z}_1 z_2\right) = 0$ in this case, so the vector product is zero.

Conversely, using polar representations 8

$$\overline{z}_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_2 - \theta_1) \tag{17}$$

$$\Im(\overline{z}_1 z_2) = r_1 r_2 \sin(\theta_2 - \theta_1) \tag{18}$$

Requiring this to be zero gives us $\theta_2 = \theta_1 + k\pi$ for integer k, so the two numbers are parallel or antiparallel. □