SERIES SUMMATION WITH RESIDUES EXAMPLES

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Here are a few examples of infinite series calculated using residues. We use the formula

$$\sum_{k=-\infty}^{\infty} f(k) = -\sum_{\ell} r_{\ell} \tag{1}$$

where r_{ℓ} is a residue of $g(z) = \pi f(z)\cot(\pi z)$ at a pole of $f(z)$. This assumes that $f(z)$ has no poles at integer real values.

Example 1. Given

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} \tag{2}$$

Here $f(z) = \frac{1}{z^2+1}$ and there are poles at $z = \pm i$. We thus need the residues of

$$g(z) = \frac{\pi\cot(\pi z)}{z^2+1} \tag{3}$$

The residues can be found using the formula

$$\begin{aligned} \text{Res}(g;i) &= \left.\frac{\pi\cot(\pi z)}{\frac{d}{dz}(z^2+1)}\right|_{z=i} &(4)\\ &= \frac{\pi\cot(i\pi)}{2i} &(5)\\ &= -\frac{\pi}{2} i\cot(i\pi) &(6)\\ &= -\frac{\pi}{2}\coth(\pi) &(7)\end{aligned}$$

where we used the identity

$$i\cot(iz) = \coth(z) \tag{8}$$

The residue at $z = -i$ turns out to be the same, so we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = -\left(-\frac{\pi}{2}\coth(\pi) - \frac{\pi}{2}\coth(\pi)\right) \tag{9}$$

$$= \pi \coth(\pi) \tag{10}$$

Example 2. Given

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \tag{11}$$

This example has a couple of features that are different from the previous example. First, the sum is over $[1,\infty]$ rather than $[-\infty,\infty]$. Second, the function $f(z) = \frac{1}{z^2}$ has a pole at $z = 0$, which is a real integer value. However, if we follow through the derivation of 1 in the earlier post, we see that the contour integral around the square is still zero, so the sum of all the residues at integer k values must still be zero. That is

$$\sum_{k=-\infty}^{-1} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^2} = -\text{Res}(g(z);0) \tag{12}$$

Since the summand is an even function of k, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{1}{2}\text{Res}(g(z);0) \tag{13}$$

To find the residue, we can use the tabulated series for $\cot(\pi z)$ (for example, see here):

$$\cot(\pi z) = \frac{1}{\pi k} - \frac{\pi k}{3} - \frac{1}{45}\pi^3 k^3 - \ldots \tag{14}$$

Multiplying this by π/z^2 to get $g(z)$, we have the Laurent series

$$\frac{\pi \cot(\pi z)}{z^2} = \frac{1}{k^3} - \frac{\pi^2}{3k} - \frac{\pi^4 k}{45} - \ldots \tag{15}$$

The residue is the coefficient of $1/k$ so we have

$$\text{Res}(g(z);0) = -\frac{\pi^2}{3} \tag{16}$$

so

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{17}$$

From 14, we can also read off

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{2}\left(\frac{\pi^4}{45}\right) = \frac{\pi^4}{90} \tag{18}$$

If we extend the Laurent series for $\cot(\pi z)$ far enough, we can read off the sum of

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \tag{19}$$

for any positive integer n.