SERIES SUMMATION WITH RESIDUES

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We can use the residue theorem to find the sums of certain infinite series.

Suppose we have a rational function

$$f(z)=\frac{P(z)}{Q(z)} \tag{1}$$

where the degree of the polynomial Q is at least 2 greater than the degree of P. We also assume that $f(z)$ has no poles at the real integer points $k=0,\pm1,\pm2,\ldots$

Now consider

$$\begin{aligned} g(z) &= \pi f(z)\cot(\pi z) && (2)\\ &= \pi f(z)\frac{\cos(\pi z)}{\sin(\pi z)} && (3)\end{aligned}$$

The function $g(z)$ has simple poles at the zeroes of $\sin(\pi z)$, that is, at $z=0,\pm1,\pm2,\ldots$ As the poles are simple, we can find the residues using the formula

$$\begin{aligned} \text{Res}(g;k) &= \frac{\pi P(k)\cos(\pi k)}{[Q(k)\sin\pi k]'} && (4)\\ &= \frac{\pi P(k)\cos(\pi k)}{Q'(k)\sin(\pi k)+Q(k)\pi\cos(\pi k)} && (5)\\ &= \frac{\pi P(k)\cos(\pi k)}{Q(k)\pi\cos(\pi k)} && (6)\\ &= f(k) && (7)\end{aligned}$$

Now consider the contour defined by the square with corners at

$$\left(N+\frac{1}{2}\right)(1+i),\left(N+\frac{1}{2}\right)(-1+i),\left(N+\frac{1}{2}\right)(-1-i),\left(N+\frac{1}{2}\right)(1-i) \tag{8}$$

where N is a non-negative integer. This is the square with top edge $z = \left(N+\frac{1}{2}\right)i$, left edge $z = -\left(N+\frac{1}{2}\right)$, bottom edge $z = -\left(N+\frac{1}{2}\right)i$ and right edge $z = \left(N+\frac{1}{2}\right)$. Now consider bounds on $\cot(\pi z)$ on this square. First, we look at the right edge, where $z = \left(N+\frac{1}{2}\right)(1+iy)$ with $-1 \le y \le 1$. We have

$$\cot\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right) = \frac{\cos\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right)}{\sin\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right)} \tag{9}$$

Expanding the trig functions we have, for the numerator

$$\cos\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right) = \cos\left(\left(N+\frac{1}{2}\right)\pi\right)\cos\left(\left(N+\frac{1}{2}\right)\pi iy\right) - \sin\left(\left(N+\frac{1}{2}\right)\pi\right)\sin\left(\left(N+\frac{1}{2}\right)\pi iy\right) \tag{10}$$

$$= 0 - (-1)^N \sin\left(\left(N+\frac{1}{2}\right)\pi iy\right) \tag{11}$$

$$= \frac{(-1)^{N+1}}{2i}\left[\exp\left(i\left(N+\frac{1}{2}\right)\pi iy\right) - \exp\left(-i\left(N+\frac{1}{2}\right)\pi iy\right)\right] \tag{12}$$

$$= \frac{(-1)^{N+1}}{2i}\left[\exp\left(-\left(N+\frac{1}{2}\right)\pi y\right) - \exp\left(\left(N+\frac{1}{2}\right)\pi y\right)\right] \tag{13}$$

Expanding the denominator gives

$$\sin\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right) = \sin\left(\left(N+\frac{1}{2}\right)\pi\right)\cos\left(\left(N+\frac{1}{2}\right)\pi iy\right) + \cos\left(\left(N+\frac{1}{2}\right)\pi\right)\sin\left(\left(N+\frac{1}{2}\right)\pi iy\right) \tag{14}$$

$$= (-1)^N \cos\left(\left(N+\frac{1}{2}\right)\pi iy\right) + 0 \tag{15}$$

$$= \frac{(-1)^N}{2}\left[\exp\left(i\left(N+\frac{1}{2}\right)\pi iy\right) + \exp\left(-i\left(N+\frac{1}{2}\right)\pi iy\right)\right] \tag{16}$$

$$= \frac{(-1)^N}{2}\left[\exp\left(-\left(N+\frac{1}{2}\right)\pi y\right)+\exp\left(\left(N+\frac{1}{2}\right)\pi y\right)\right] \tag{17}$$

As $N \to \infty$, the exponentials with positive exponents in 13 and 17 dominate, so the magnitude of 9 tends to

$$\lim_{N\to\infty}\left|\cot\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right)\right| = \lim_{N\to\infty}\left|\frac{\frac{(-1)^{N+1}}{2i}\left[\exp\left(-\left(N+\frac{1}{2}\right)\pi y\right)-\exp\left(\left(N+\frac{1}{2}\right)\pi y\right)\right]}{\frac{(-1)^N}{2}\left[\exp\left(-\left(N+\frac{1}{2}\right)\pi y\right)+\exp\left(\left(N+\frac{1}{2}\right)\pi y\right)\right]}\right| \tag{18}$$

$$= \lim_{N\to\infty}\frac{\exp\left(\left(N+\frac{1}{2}\right)\pi y\right)}{\exp\left(\left(N+\frac{1}{2}\right)\pi y\right)} \tag{19}$$

$$= 1 \tag{20}$$

A similar argument gives the same bound on the left edge, where $z = -1 + iy$.

For the top edge, we have $z = \left(N+\frac{1}{2}\right)(x+i)$, with $-1 \le x \le 1$. In this case we have

$$\cot\left(\pi\left(N+\frac{1}{2}\right)(1+iy)\right) = \frac{\cos\left(\pi\left(N+\frac{1}{2}\right)(x+i)\right)}{\sin\left(\pi\left(N+\frac{1}{2}\right)(x+i)\right)} \tag{21}$$

Expanding the numerator:

$$\cos\left(\pi\left(N+\frac{1}{2}\right)(x+i)\right) = \cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\cos\left(\pi\left(N+\frac{1}{2}\right)i\right) - \sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\sin\left(\pi\left(N+\frac{1}{2}\right)i\right) \tag{22}$$

Take the first term on the RHS:

$$\cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\cos\left(\pi\left(N+\frac{1}{2}\right)i\right) = \cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\times \frac{1}{2}\left[\exp\left(i\pi\left(N+\frac{1}{2}\right)i\right)+\exp\left(-i\pi\left(N+\frac{1}{2}\right)i\right)\right] \tag{23}$$

$$= \frac{1}{2}\cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\times$$

$$\left[\exp\left(-\pi\left(N+\frac{1}{2}\right)\right)+\exp\left(\pi\left(N+\frac{1}{2}\right)\right)\right] \tag{24}$$

In this expression, the first cosine factor is bounded by $\pm\frac{1}{2}$ since it's the cosine of a real number. In the two exponentials, the second term will dominate for large N.

In the second term, we have

$$\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\sin\left(\pi\left(N+\frac{1}{2}\right)i\right)=\frac{1}{2i}\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\times \left[\exp\left(-\pi\left(N+\frac{1}{2}\right)\right)-\exp\left(\pi\left(N+\frac{1}{2}\right)\right)\right] \tag{25}$$

Again, the sine factor is bounded by $\pm\frac{1}{2}$ and the sum of exponentials is dominated by the second term $\exp\left(\pi\left(N+\frac{1}{2}\right)\right)$ for large N. We can thus write the numerator as

$$\cos\left(\pi\left(N+\frac{1}{2}\right)(x+i)\right)=\exp\left(\pi\left(N+\frac{1}{2}\right)\right)\left[\frac{1}{2}\cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\left(\exp\left(-2\pi\left(N+\frac{1}{2}\right)\right)+\frac{1}{2i}\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\left(\exp\left(-2\pi\left(N+\frac{1}{2}\right)\right)-1\right)\right] \tag{26}$$

Expanding the denominator of 21, we have

$$\sin\left(\pi\left(N+\frac{1}{2}\right)(x+i)\right)=\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\cos\left(\pi\left(N+\frac{1}{2}\right)i\right)+\cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\sin\left(\pi\left(N+\frac{1}{2}\right)i\right) \tag{27}$$

$$=\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\times \frac{1}{2}\left[\exp\left(i\pi\left(N+\frac{1}{2}\right)i\right)+\exp\left(-i\pi\left(N+\frac{1}{2}\right)i\right)\right]+ \cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\times$$

$$\frac{1}{2i}\left[\exp\left(i\pi\left(N+\frac{1}{2}\right)i\right)-\exp\left(-i\pi\left(N+\frac{1}{2}\right)i\right)\right] \tag{28}$$

$$=\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\frac{1}{2}\left[\exp\left(-\pi\left(N+\frac{1}{2}\right)\right)+\exp\left(\pi\left(N+\frac{1}{2}\right)\right)\right]+$$
$$\cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\frac{1}{2i}\left[\exp\left(-\pi\left(N+\frac{1}{2}\right)\right)-\exp\left(\pi\left(N+\frac{1}{2}\right)\right)\right] \tag{29}$$

$$=\exp\left(\pi\left(N+\frac{1}{2}\right)\right)\times$$
$$\left[\frac{1}{2}\sin\left(\pi\left(N+\frac{1}{2}\right)x\right)\left(\exp\left(-2\pi\left(N+\frac{1}{2}\right)\right)+1\right)+\right.$$
$$\left.\frac{1}{2i}\cos\left(\pi\left(N+\frac{1}{2}\right)x\right)\left(\exp\left(-2\pi\left(N+\frac{1}{2}\right)\right)-1\right)\right] \tag{30}$$

Thus both the numerator and denominator are dominated by the factor $\exp\left(\pi\left(N+\frac{1}{2}\right)\right)$, so in the limit of $N\to\infty$ these terms will cancel out and we're left with a finite bound, independent of N.

We can now find a limit as $N\to\infty$ for the contour integral of 2 around the square. Since the degree n of Q is at least 2 greater than the degree m of P, the magnitude of $f(z)$ goes as $1/N^{n-m}$ as the square expands with increasing N. However, the perimeter of the square is $4s$ where s is the side length, which is $2N+1$. Thus the contour integral is bounded by

$$\left|\oint_C \pi f(z)\cot(\pi z)\,dz\right| \le \frac{4(2N+1)}{N^{n-m}} \to 0 \tag{31}$$

Now for the clever bit. This contour integral is, by the residue theorem

$$\oint_C \pi f(z)\cot(\pi z)\,dz = 2\pi i\sum[\text{residues}] \tag{32}$$

We saw from 7 that the function $\pi f(z)\cot(\pi z)$ has a residue of value $f(k)$ at every integer value k. Furthermore, since $Q(z)$ (the denominator of f) is a polynomial, it has a finite number of roots, which we've assumed do not occur for any integer values of z, which means that the function $\pi f(z)\cot(\pi z)$ will also have poles at the poles of $f(z)$. This means that

$$\oint_C \pi f(z)\cot(\pi z)\,dz = 2\pi i\left(\sum_{k=-\infty}^{\infty} f(k)+\sum_{\ell} r_\ell\right)=0 \tag{33}$$

where r_ℓ is a residue of $g(z) = \pi f(z)\cot(\pi z)$ at a pole of $f(z)$. In other words

$$\boxed{\sum_{k=-\infty}^{\infty} f(k) = -\sum_{\ell} r_\ell} \tag{34}$$

Example 1. Find

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} \tag{35}$$

Here, we take

$$f(k) = \frac{1}{\left(k-\frac{1}{2}\right)^2} \tag{36}$$

As f has a pole of order 2 at $k = \frac{1}{2}$, we need the residue of g at this point. That is

$$\text{Res}\left(\pi\frac{\cot(\pi k)}{\left(k-\frac{1}{2}\right)^2}, \frac{1}{2}\right) = \frac{d}{dk}\left[\pi\left(k-\frac{1}{2}\right)^2 f(k)\cot(\pi k)\right]\Bigg|_{k=\frac{1}{2}} \tag{37}$$

$$= \pi^2\left(-1-\cot(\pi k)^2\right)\Big|_{k=\frac{1}{2}} \tag{38}$$

$$= -\pi^2 \tag{39}$$

Therefore

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)^2} = \pi^2 \tag{40}$$

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