SETS IN THE COMPLEX PLANE - TERMINOLOGY

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Sets of complex numbers can be represented as portions of the complex plane. There is rather a lot of terminology associated with such sets, so we'll list those here, along with some examples.

A set consisting of a circular disk of points that does not include the boundary of the circle is an *open disk*. The set $|z| < 4$ represents an open disk of radius 4 centred at the origin. open disk

An *interior point* z_0 of a set S is a point which is the centre of an open disk that is entirely contained within S. All points satisfying $|z| < 4$ are interior points, as no matter how close we get to the circle $|z| = 4$, there is always an open disk around such points that is contained within the open disk. A point actually on the circle $|z| = 4$ is not an interior point, since any open disk centred on the boundary circle has points both within and outside of S. An open disk around a point z_0 is called the *neighbourhood* of z_0. interior point neighbourhood

Following on from this, a set all of whose points are interior points is called an *open set*. Thus $|z| < 4$ is an open set, but $|z| \leq 4$ is not, since the latter contains points on the bounding circle. open set

A *polygonal path* is a path consisting of a number of linked straight line segments lying entirely within the set S. A set is called *connected* if there is at least one polygonal path connecting any two points in the set. The set $|z| < 4$ is connected. It is not necessary for there to be a *single* line segment connecting any two points; all that is required is that we can draw a polygonal path between any two points without going outside the set. For example, the ring defined by $2 < |z| < 4$ is connected even though the circular disk $|z| \leq 2$ is excluded, since we can draw a polygon path entirely within the ring. polygonal path connected

An open connected set is called a *domain*. The sets $|z| < 4$ and $2 < |z| < 4$ are both domains, but the sets $|z| \leq 4$ and $2 < |z| \leq 4$ are not, since neither is open. domain

A point z_0 is called a *boundary point* of a set S if every neighbourhood of z_0 contains at least one point in S and one point not in S. boundary point

A set is called *closed* if it contains all its boundary points. The point $z_0 = 4i$ is a boundary point of both $|z| < 4$ and $|z| \leq 4$, but the first set is open while the second is closed. The complement of a closed set (that is, closed all points not in the closed set) is an open set. Note that it is possible for a set to be neither open nor closed (Example 4 below).

A set S is *bounded* if there is a real number R such that $|z| < R$ for every point in S. This means that there is a circle large enough to contain all the points in S. The set $|z| < 4$ is bounded; the set $|z| > 4$ is not.

Finally, a *region* is a domain (open connected set) together with some, none or all of its boundary points. Thus every domain is a region. region

We'll now look at a few examples.

Example 1. $|z-1+i| \leq 3$. This defines a circular set centred at $1-i$ of radius 3. It is closed, as it contains its boundary circle. As it is closed, it is not a domain, but it is connected and is therefore a region. Since it contains all its boundary points, it is a closed region.

Example 2. $|\text{Arg } z| < \frac{\pi}{4}$. This refers to the principal argument since Arg is written with a capital A. Thus the argument lies in the open range $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. This consists of numbers lying between the lines $y = x$ and $y = -x$, with $x > 0$ (recall the argument is undefined for $z = 0$). There is no restriction on the magnitude of z, so this is an unbounded set. It is open, as it does not include its boundary lines. It is connected and is therefore also a region.

Example 3. $0 < |z-2| < 3$. This is a circular disk centred at $(2,0)$ with radius 3. It excludes both the origin and the boundary and so is an open set. It's also connected as we can find a polygonal path connecting any two points (we can always avoid going through the origin) and is therefore a region. It's also bounded.

Example 4. $-1 < \Im z \leq 1$. This is the horizontal strip between $y = -1$ and $y = 1$, with the lower boundary excluded, but with the upper boundary included. As any point on the line $y = 1$ is in the set, it is not open, but as the lower boundary is *not* included, it is not closed either. It is still a region, as a region is allowed to contain some of the boundary point. As the real part of z can be any value, it is not bounded.

Example 5. $|z| \geq 2$. This defines all points outside the circle centred at the origin with radius 2, but including the circle itself as the boundary, thus it is a closed set as there is no outer boundary. It is connected as we can always find a polygonal path around the circle. It's unbounded, and is also a region as it is allowed to contain its boundary.

Example 6. $(\Re z)^2 > 1$. This consists of the infinite half-plane with $\Re z < -1$ and the other infinite half-plane with $\Re z > 1$. Boundaries are excluded, so this is an open set. However, it is not connected, as there is no path between the half-planes that does not cross the area between the planes. Thus it is not a domain or a region. It's also unbounded.

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