SMOOTH CURVES

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When we do single integration with real functions, the range of integration is specified by the limits on the integral. Since the space of real variables is one-dimensional, we are effectively integrating over some portion of the real axis. When we come to integration with complex variables, however, we cannot just specify the limits of integration, since for any two points in the complex plane, there are an infinite number of possible paths connecting them. Integration with complex variables therefore requires the specification of a path, known as a *contour*, over which the integration is done.

As a first step in defining a contour, we consider the idea of a *smooth curve*. Saff and Snider's book visualizes the creation of a smooth curve as the process of an artist drawing a curve on paper. The idea is that a smooth curve can be drawn without lifting the pen off the paper, and with a finite non-zero velocity at every point in the drawing process. We can think of the process by specifying the coordinates of the pen as functions of time t, so we have

$$z(t) = x(t) + iy(t) \tag{1}$$

Here, both $x(t)$ and $y(t)$ are continuous functions, and there is no value t_0 of t for which $x'(t_0) = y'(t_0) = 0$. That is, $z'(t) = x'(t) + iy'(t) \neq 0$ at any value of t.

Thus our first goal is to find a parametric form for a given smooth curve. The conditions are laid out in the formal definition.

Definition 1. A point set γ in the complex plane is called a *smooth arc* if it is the range of a continuous complex-valued function $z(t)$, for $a \le t \le b$ (where a and b are real constants, and t is a real parameter), where the following conditions are satisfied.

1. $z(t)$ has a continuous derivative on $[a,b]$.
2. $z'(t)$ is never zero on $[a,b]$. (See Example 6 below.)
3. $z(t)$ is one-to-one on $[a,b]$. This condition ensures that the endpoints of the arc are distinct points.

Definition 2. A point set γ is a *smooth closed curve* if it satisfies conditions 1 and 2 of Definition 1, but instead of condition 3, we have

3'. $z(t)$ is one-to-one on the interval $[a,b)$, but $z(a) = z(b)$ and $z'(a) = z'(b)$. That is, the start and end points are the same, and join smoothly without any corners.

Example 1. Parametrize the line segment from $1+i$ to $-2-3i$. A line segment from the point z_1 to z_2 can be parametrized in the form

$$z(t) = z_1 + (z_2 - z_1)t \tag{2}$$

where $0 \leq t \leq 1$. In this case we have

$$z(t) = 1 + i + (-3-4i)t \tag{3}$$

This is a smooth arc.

Example 2. Parametrize the circle $|z-2i| = 4$ traversed once in the clockwise direction, starting at $z = 4+2i$. A circle of radius r centred at the origin can be written as

$$z = re^{i\theta} \tag{4}$$

where $0 \leq \theta \leq 2\pi$. However, this traverses the circle in the counterclockwise direction. To reverse the direction, we invert the exponent, so that

$$z = re^{-i\theta} \tag{5}$$

To shift the centre, we add the centre's coordinates, so we have

$$z = 2i + 4e^{-i\theta} \tag{6}$$

The starting point of $z = 4+2i$ lies on the horizontal line through the circle's centre, so corresponds to $\theta = 0$. Thus 6 is the required parametrization. This is a smooth closed curve.

Example 3. The arc of the circle $|z| = R$ in the second quadrant, between $z = Ri$ and $z = -R$. This circle is centred at the origin, so we have

$$z = Re^{i\theta} \tag{7}$$

with $\frac{\pi}{2} \leq \theta \leq \pi$. Alternatively, if we want θ to run between 0 and 2π, we could write

$$z = Re^{i(\theta+2\pi)/4} \tag{8}$$

This is a smooth arc.

Example 4. The arc of a parabola $y = x^2$ between $(1,1)$ and $(3,9)$. In this case, the real part is x and the imaginary part is y, so we have

$$z(x) = x + iy^2 \tag{9}$$

for $1 \leq x \leq 3$. This is another smooth arc.

Example 5. The ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{10}$$

is a smooth closed curve. We can parametrize it by writing

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned} \tag{11}$$

so the required form is

$$z(\theta) = a\cos\theta + ib\sin\theta \tag{12}$$

with $0 \leq \theta \leq 2\pi$.

Example 6. To see why we need to invoke condition 2 in Definition 1 above, that is, that we require the derivative $z'(t)$ to be non-zero, consider the parametrized curve

$$z(t) = t^2 + it^3 \tag{13}$$

with $-1 \leq t \leq 1$. The plot of the curve is in Fig. 1. We see that it has a cusp, or corner, at $t = 0$. The derivative is

$$z'(t) = 2t + 3it^2 \tag{14}$$

which is zero at $t = 0$. Using the artist analogy above, this corresponds to a point where the artist pauses (the velocity of drawing becomes zero) because of an abrupt change of direction in the line, which we're prohibiting in our definition of a smooth curve.

Example 7. Not every parametrization with a zero derivative represents a non-smooth curve. For example, consider

$$z(t) = t^3 + it^6 \tag{15}$$

for $-1 \leq t \leq 1$. The derivative is

$$z'(t) = 3t^2 + 6it^5 \tag{16}$$

which has a zero a $t = 0$. However, if we plot this curve we get Fig. 2, which appears smooth.

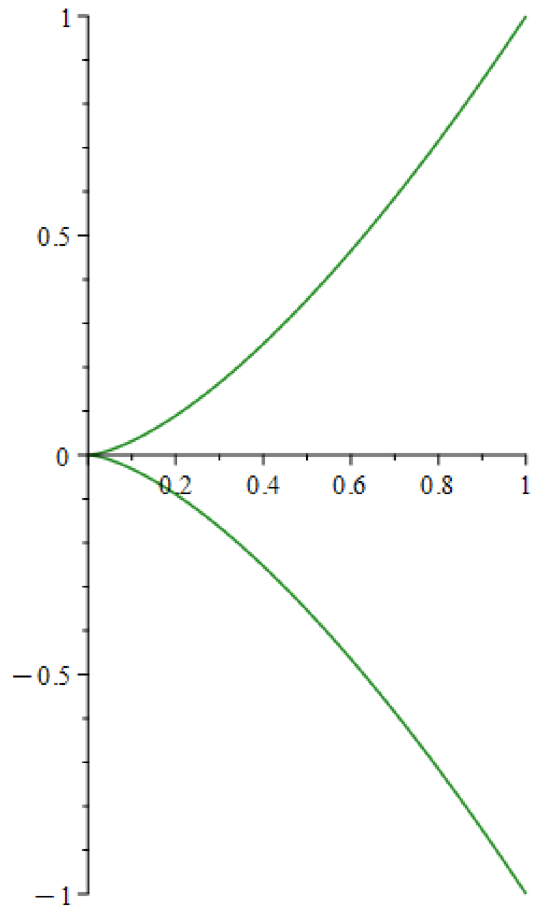

FIGURE 1. The curve $z(t) = t^2 + it^3$.

An admissible parametrization of this curve is

$$z(t) = t + it^2 \tag{17}$$

which does not have a zero derivative. All that matters is that the imaginary part is the square of the real part, so this curve is really just the parabola $y = x^2$. Sometimes this can result in an inadmissible parametrization, but for a smooth curve, there should always be at least one parametrization that is admissible, that is, that doesn't have zero derivative.

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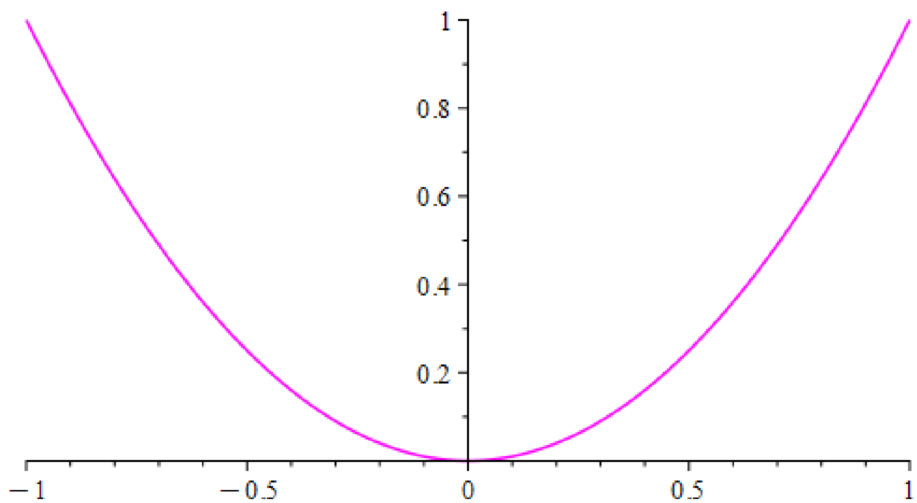

FIGURE 2. The curve $z(t) = t^3 + it^6$.