STEREOGRAPHIC PROJECTION AND THE RIEMANN SPHERE

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Post date: 19 November 2024.

A *stereographic projection* is the projection of a point on a plane onto a sphere of radius 1. The sphere is centred at the origin and its equatorial plane coincides with the plane being projected onto the sphere. The projection is done by drawing a line from the north pole of the sphere (with coordinates $(0,0,1)$) to the point on the plane. The projection point is the point where the line intersects the sphere. By this means, we see that points on the plane that lie outside the sphere's equator project onto points in the northern hemisphere of the sphere, while points on the plane that lie inside the sphere project onto the southern hemisphere. The origin of the plane projects onto the south pole of the sphere. There is no point on the plane that projects onto the north pole (or, alternatively, the north pole represents points at infinity on the plane). The sphere is called the *Riemann sphere*.

If we take the plane to be the complex plane, then we seek equations that give the coordinates (x_1,x_2,x_3) of the point on the sphere that result from projecting the point z in the complex plane. We can do this as follows.

Suppose that we write $z=(x,y,0)$. Then we seek the line that contains the north pole $(0,0,1)$ and the point $(x,y,0)$. In parametric form, we can represent the line by the sum of a vector that points to a point on the line and a multiple of a vector parallel to the line. We take the fixed point to be $(0,0,1)$ and the vector parallel to the line as $(1,0,0)-(x,y,0)$. Then we have the points on the line given by

$$(0,0,1)-t\left[(1,0,0)-(x,y,0)\right]=[tx,ty,1-t] \tag{1}$$

where t is the parameter that varies from 0 to 1, giving the line that extends from $(0,0,1)$ to $(x,y,0)$. That is, we have

$$\begin{aligned} x_1 &= tx \\ x_2 &= ty \\ x_3 &= 1-t \end{aligned} \tag{2}$$

The sphere has the equation

$$x_1^2+x_2^2+x_3^2=1 \tag{3}$$

so we seek the intersection of 2 and 3, given by

$$x_1^2+x_2^2+x_3^2=1=(tx)^2+(ty)^2+(1-t)^2 \tag{4}$$

Simplifying, we have

$$t^2\left(x^2+y^2+1\right)-2t+1=1 \tag{5}$$

$$t\left[\left(x^2+y^2+1\right)t-2\right]=0 \tag{6}$$

The two solutions are $t=0$, which corresponds to the north pole, and

$$t=\frac{2}{x^2+y^2+1}=\frac{2}{|z|^2+1} \tag{7}$$

We substitute this into 2 to get

$$\begin{aligned} x_1 &= \frac{2\Re z}{|z|^2+1} \\ x_2 &= \frac{2\Im z}{|z|^2+1} \\ x_3 &= 1-\frac{2}{|z|^2+1}=\frac{|z|^2-1}{|z|^2+1} \end{aligned} \tag{8}$$

Example 1. Projection of i onto the Riemann sphere. From 8 we have

$$\begin{aligned} x_1 &= 0 \\ x_2 &= \frac{2}{1+1}=1 \\ x_3 &= \frac{1-1}{1+1}=0 \end{aligned} \tag{9}$$

Thus i projects onto the equator on the positive x_2 axis.

Example 2. Project $6-8i$. We have

$$|z|=\sqrt{36+64}=10 \tag{10}$$

Since $|z|>1$, the projection is onto the northern hemisphere.

$$
\begin{aligned}
x_1 &= \frac{2\times 6}{100+1} = \frac{12}{101} \\
x_2 &= \frac{2\times(-8)}{101} = -\frac{16}{101} \\
x_3 &= \frac{99}{101}
\end{aligned}
\tag{11}
$$

The projection is close to the north pole.

Example 3. Project $-\frac{3}{10}+\frac{2}{5}i$. We have

$$
|z| = \sqrt{\frac{9}{100}+\frac{4}{25}} = \frac{1}{2} \tag{12}
$$

Since $|z|<1$, the projection is onto the southern hemisphere.

$$
\begin{aligned}
x_1 &= \frac{-3/5}{5/4} = -\frac{12}{25} \\
x_2 &= \frac{4/5}{5/4} = \frac{16}{25} \\
x_3 &= \frac{1/4-1}{1/4+1} = -\frac{3}{5}
\end{aligned}
\tag{13}
$$

Example 4. Find the projection of the point $1/\overline{z}$. We have

$$
\frac{1}{\overline{z}} = \frac{z}{|z|^2} \tag{14}
$$

so

$$
\begin{aligned}
\Re\left(\frac{1}{\overline{z}}\right) &= \frac{x}{|z|^2} \\
\Im\left(\frac{1}{\overline{z}}\right) &= \frac{y}{|z|^2} \\
\left|\frac{1}{\overline{z}}\right| &= \frac{1}{|z|}
\end{aligned}
\tag{15}
$$

Therefore, if the projection of z is (x_1,x_2,x_3) and the projection of $1/\overline{z}$ is $\left(x'_1,x'_2,x'_3\right)$ we have

$$\begin{aligned}
x_1' &= \frac{2x/|z|^2}{1+1/|z|^2} = \frac{2x}{|z|^2+1} = x_1 \\
x_2' &= \frac{2y/|z|^2}{1+1/|z|^2} = \frac{2y}{|z|^2+1} = x_2 \\
x_3' &= \frac{1/|z|^2-1}{1/|z|^2+1} = \frac{1-|z|^2}{1+|z|^2} = -x_3
\end{aligned} \tag{16}$$

Thus the projection of $1/\overline{z}$ is the reflection of the projection of z in the equatorial plane.

Example 5. Find the projection of $-1/\overline{z}$. This is just the negative of the previous example, so we have, with the new projection being $\left(x_1'', x_2'', x_3''\right)$:

$$\begin{aligned}
x_1'' &= -x_1' = -x_1 \\
x_2'' &= -x_2' = -x_2 \\
x_3'' &= -x_3' = x_3
\end{aligned} \tag{17}$$

The projection of $-1/\overline{z}$ has the same x_3 as the projection of z, but is diametrically opposite in the horizontal plane containing (x_1, x_2, x_3). This isn't a reflection through the origin.

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