STEREOGRAPHIC PROJECTION OF CIRCLES

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An interesting feature of stereographic projections is that every circle on the Riemann sphere is the projection of either a line or a circle in the complex plane. Saff and Snider prove the inverse of this in their book. That is they start from the equations for a circle or line in the complex plane and show that they project onto circles on the Riemann sphere. Here, we'll show that starting from a circle on the Riemann sphere, it is the projection of a circle or line in the complex plane.

We begin by noting that any circle on the Riemann sphere is the intersection of a plane in 3d space with the sphere. The general equation for a 3d plane is

$$Cx_1+Dx_2+(A-E)\,x_3+A+E=0 \tag{1}$$

where A,B,C,D,E are constants and (x_1,x_2,x_3) are the 3d coordinates. We write the constants as $A-E$ and $A+E$ to make the following proof easier. We could find the equation of the circle by setting 1 equal to the Riemann sphere condition $x_1^2+x_2^2+x_3^2=1$,[1] but our intention here is to show that this projection corresponds to a circle or line in the complex plane.

Rearranging 1 we have

$$A\,(1+x_3)+Cx_1+Dx_2+E\,(1-x_3)=0 \tag{2}$$

Multiplying this through by $(1-x_3)$ we have

$$A\left(1-x_3^2\right)+Cx_1\,(1-x_3)+Dx_2\,(1-x_3)+E\,(1-x_3)^2=0 \tag{3}$$

Since (x_1,x_2,x_3) is on the Riemann sphere, which is a unit sphere, we have

$$x_1^2+x_2^2+x_3^2=1 \tag{4}$$

so 3 becomes

[1] I worked this out using Maple, but the general expression is very messy. For the special case $A=1,C=D=0,E=-1$, which gives the plane $x_3=0$, though, we do get the equator of the sphere as required.

$$A\left(x_1^2+x_2^2\right)+Cx_1\left(1-x_3\right)+Dx_2\left(1-x_3\right)+E\left(1-x_3\right)^2=0 \qquad (5)$$

We can now divide by $\left(1-x_3\right)^2$ to get

$$A\left[\left(\frac{x_1}{1-x_3}\right)^2+\left(\frac{x_2}{1-x_3}\right)^2\right]+\frac{Cx_1}{1-x_3}+\frac{Dx_2}{1-x_3}+E=0 \qquad (6)$$

We now recall the parametric form of the line that projects onto (x_1,x_2,x_3):

$$\begin{aligned} x_1 &= tx \\ x_2 &= ty \\ x_3 &= 1-t \end{aligned} \qquad (7)$$

where (x,y) is the point in the complex plane with $z=x+iy$. Eliminating t from 7 we have

$$\begin{aligned} t &= 1-x_3 \\ x &= \frac{x_1}{1-x_3} \\ y &= \frac{x_2}{1-x_3} \end{aligned} \qquad (8)$$

Substituting this into 6 we an eliminate the projected coordinates in favour of (x,y). We have

$$A\left(x^2+y^2\right)+Cx+Dy+E=0 \qquad (9)$$

The general equation for a circle in the plane is

$$\left(x-x_0\right)^2+\left(y-y_0\right)^2=r^2 \qquad (10)$$

where (x_0,y_0) is the centre of the circle and r is its radius. Expanding this gives

$$x^2+y^2-2xx_0-2yy_0+x_0^2+y_0^2-r^2=0 \qquad (11)$$

Comparing this with 9 we see that 9 is the general equation for a circle if we divide through by A, with

$$\begin{aligned}\frac{C}{A} &= -2x_0\\ \frac{D}{A} &= -2y_0\\ \frac{E}{A} &= x_0^2+y_0^2-r^2\end{aligned} \tag{12}$$

That is, a circle on the Riemann sphere does indeed arise from mapping a circle from the complex plane.

Note that if $A=0$ in 9, we get the equation of a line $Cx+Dy+E=0$. This corresponds to the circle defined by the intersection of the plane 1 where $A=0$ with the Riemann sphere. That is, it is the intersection of

$$Cx_1+Dx_2-Ex_3+E=0 \tag{13}$$

with

$$x_1^2+x_2^2+x_3^2=1 \tag{14}$$

For example, if $D=E=0$, we have the plane $x_1=0$ (the x_2x_3 plane), which corresponds to the circle $x_2^2+x_3^2=1$ on the sphere. This is a vertical circle, and maps to the line that is the y axis in the complex plane.

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