SUMS OF MULTIPLE ANGLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 12 November 2024.

We can use De Moivre's formula to find sums of multiple-angle trig functions. De Moivre's formula states

$$(\cos\theta+i\sin\theta)^{n}=\cos(n\theta)+i\sin(n\theta) \tag{1}$$

We can use this to find the sums

$$\sum_{k=0}^{n}\cos(k\theta) \tag{2}$$

$$\sum_{k=1}^{n}\sin k\theta \tag{3}$$

We also need the formula for the sum of a geometric series:

$$\sum_{k=0}^{n}z^{k}=\frac{z^{n+1}-1}{z-1} \tag{4}$$

First, consider the sum of cosines. We have

$$S\equiv\sum_{k=0}^{n}\left[\cos(k\theta)+i\sin(k\theta)\right]=\sum_{k=0}^{n}(\cos\theta+i\sin\theta)^{k} \tag{5}$$

$$=\frac{(\cos\theta+i\sin\theta)^{n+1}-1}{(\cos\theta+i\sin\theta)-1} \tag{6}$$

$$=\frac{\cos((n+1)\theta)+i\sin((n+1)\theta)-1}{\cos\theta+i\sin\theta-1} \tag{7}$$

To find the sum of cosines, we need the real part of this. This is a somewhat messy calculation, so I used Maple to simplify the result. We need to eliminate the imaginary part from the denominator, so we multiply 7 by

$$\frac{\cos\theta-i\sin\theta-1}{\cos\theta-i\sin\theta-1} \tag{8}$$

After collecting terms and simplifying, we get

$$\Re(S) = \frac{1-\cos\theta+\cos n\theta-\cos((n+1)\theta)}{2(1-\cos\theta)} \tag{9}$$

$$= \frac{1}{2}+\frac{\cos n\theta-\cos((n+1)\theta)}{2(1-\cos\theta)} \tag{10}$$

This can be converted into a formula involving half-angles. To convert the denominator, we use the half-angle formula

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \tag{11}$$

This gives

$$\Re(S) = \frac{1}{2}+\frac{\cos n\theta-\cos((n+1)\theta)}{4\sin^2\theta/2} \tag{12}$$

To convert the numerator, we use the formulas

$$\cos((n+1)\theta) = \cos\left[\left(n+\frac{1}{2}+\frac{1}{2}\right)\theta\right] \tag{13}$$

$$= \cos\left[\left(n+\frac{1}{2}\right)\theta\right]\cos\frac{\theta}{2}-\sin\left[\left(n+\frac{1}{2}\right)\theta\right]\sin\frac{\theta}{2} \tag{14}$$

$$\cos n\theta = \cos\left[\left(n+\frac{1}{2}-\frac{1}{2}\right)\theta\right] \tag{15}$$

$$= \cos\left[\left(n+\frac{1}{2}\right)\theta\right]\cos\frac{\theta}{2}+\sin\left[\left(n+\frac{1}{2}\right)\theta\right]\sin\frac{\theta}{2} \tag{16}$$

Combining these we get

$$\cos n\theta-\cos((n+1)\theta) = 2\sin\left[\left(n+\frac{1}{2}\right)\theta\right]\sin\frac{\theta}{2} \tag{17}$$

Plugging this into 12 we get the final form

$$\boxed{\sum_{k=0}^{n}\cos k\theta = \frac{1}{2}+\frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{\sin\theta/2}} \tag{18}$$

To get the sum of sines, we find the imaginary part of 7. Again, using Maple to simplify things, we get (the sum starts at $k=1$ since the $k=0$ term gives zero anyway):

$$\Im S = \sum_{k=1}^{n} \sin k\theta = \frac{\sin\theta + \sin n\theta - \sin((n+1)\theta)}{2 - 2\cos\theta} \tag{19}$$

$$= \frac{\sin\theta + \sin n\theta - \sin((n+1)\theta)}{4\sin^2\theta/2} \tag{20}$$

where we used 11 to convert the denominator.

To convert the numerator to half-angle formulas, we start with

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \tag{21}$$

We apply this to all three terms in the numerator of 20, so we have

$$\sin\theta + \sin n\theta - \sin((n+1)\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - 2\sin\left(\frac{(n+1)\theta}{2}\right)\cos\left(\frac{(n+1)\theta}{2}\right) - \tag{22}$$

$$2\sin\left(\frac{\theta n}{2}\right)\cos\left(\frac{\theta n}{2}\right) \tag{23}$$

From here, we use the angle-sum formulas to expand the second term on the RHS. That is,

$$\sin\left(\frac{(n+1)\theta}{2}\right) = \sin\left[\frac{n\theta}{2} + \frac{\theta}{2}\right] \tag{24}$$

$$= \sin\frac{n\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{n\theta}{2} \tag{25}$$

$$\cos\left(\frac{(n+1)\theta}{2}\right) = \cos\left[\frac{n\theta}{2} + \frac{\theta}{2}\right] \tag{26}$$

$$= \cos\frac{n\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{n\theta}{2} \tag{27}$$

Using Maple to do the algebra, we arrive at

$$\sin\theta + \sin n\theta - \sin((n+1)\theta) = 4\sin\left(\frac{\theta n}{2}\right)\left(\sin\left(\frac{\theta n}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta n}{2}\right)\right)\sin\left(\frac{\theta}{2}\right) \tag{28}$$

$$= 4\sin\left(\frac{\theta n}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin\left[\frac{(n+1)\theta}{2}\right] \tag{29}$$

Plugging into 20 we get

$$\boxed{\sum_{k=1}^{n} \sin k\theta = \frac{\sin\left(\frac{\theta n}{2}\right) \sin\left[\frac{(n+1)\theta}{2}\right]}{\sin\left(\frac{\theta}{2}\right)}} \tag{30}$$