SUMS, PRODUCTS AND RECIPROCALS OF COMPLEX NUMBERS

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The sum of two complex numbers z_1 and z_2 is obtained by adding their real parts, and separately, their imaginary parts. That is

$$z_1+z_2=(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2) \tag{1}$$

The product is found by expanding the product of two binomials.

$$z_1z_2=(x_1+iy_1)(x_2+iy_2) \tag{2}$$
$$=x_1x_2-y_1y_2+i(x_1y_2+y_1x_2) \tag{3}$$

Given these definitions, we can see that, for addition (where the symbol $\Re$ means 'real part' and $\Im$ means 'imaginary part'):

$$\sum_{j=1}^{n}z_j=\sum_{j=1}^{n}\Re(z_j)+\sum_{j=1}^{n}\Im(z_j) \tag{4}$$

Thus

$$\Re\left(\sum_{j=1}^{n}z_j\right)=\sum_{j=1}^{n}\Re(z_j) \tag{5}$$
$$\Im\left(\sum_{j=1}^{n}z_j\right)=\sum_{j=1}^{n}\Im(z_j) \tag{6}$$

The corresponding statement for products is *not* true, however. That is

$$\Re\left(\prod_{j=1}^{n}z_j\right)\neq\prod_{j=1}^{n}\Re(z_j) \tag{7}$$
$$\Im\left(\prod_{j=1}^{n}z_j\right)\neq\prod_{j=1}^{n}\Im(z_j) \tag{8}$$

This is verified by looking at the case 3 of the product of two numbers, where we see that

$$\Re(z_1z_2) = x_1x_2 - y_1y_2 \neq \Re z_1 \times \Re z_2 = x_1x_2 \tag{9}$$

$$\Im(z_1z_2) = x_1y_2 + y_1x_2 \neq \Im z_1 \times \Im z_2 = y_1y_2 \tag{10}$$

Example 1. Suppose we have two complex numbers z_1 and z_2 such that both $z_1 + z_2$ and z_1z_2 are negative real numbers. What are the possible values for z_1 and z_2?

We have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{11}$$

We must therefore have

$$x_1 + x_2 < 0 \tag{12}$$

$$y_1 = -y_2 \tag{13}$$

From the product, we have

$$z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2) \tag{14}$$

so

$$x_1x_2 - y_1y_2 < 0 \tag{15}$$

$$x_1y_2 = -y_1x_2 \tag{16}$$

From 13, the last condition becomes

$$-x_1y_1 = -y_1x_2 \tag{17}$$

Thus either $x_1 = x_2$ or $y_1 = 0$. If we take $x_1 = x_2$ then, from 15 and 13

$$x_1^2 + y_1^2 < 0 \tag{18}$$

Since both x_1 and y_1 are real (non-zero) numbers, this is not possible, so we must have

$$y_1 = -y_2 = 0 \tag{19}$$

That is, both z_1 and z_2 must be real numbers.

Example 2. We can also derive a property of the reciprocal of a complex number. Suppose $z = x + iy$. Then

$$\frac{1}{z} = \frac{1}{x+iy} \tag{20}$$

$$= \frac{1}{x+iy}\frac{x-iy}{x-iy} \tag{21}$$

$$= \frac{x-iy}{x^2+y^2} \tag{22}$$

Since the denominator is positive, we see that the real part of a reciprocal of z has the same sign as $\Re z$, while the imaginary part has the opposite sign as $\Im z$.