TAYLOR FORM OF POLYNOMIALS

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A polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n \tag{1}$$

can be recentered at a different point z_0 and rewritten as

$$p(z) = d_0 + d_1 (z - z_0) + d_2 (z - z_0)^2 + \ldots + d_n (z - z_0)^n \tag{2}$$

Since 1 and 2 represent the same polynomial, we can express the constants d_k in terms of the a_k. A brute force way of doing this is to expand 2, collect the terms for each power of z and equate the coefficients of each z^k in the two forms. This would give us a system of $n+1$ linear equations in the $n+1$ unknowns d_k, which we would then have to solve in the traditional way.

There is in fact a much simpler way of finding the d_k in terms of the a_k. First, we note that $p(z_0) = d_0$. Next, we take the first derivative of $p(z)$ in the form 2 and evaluate this at $z = z_0$. This gives

$$p'(z_0) = d_1 \tag{3}$$

since all the terms with d_k with $k > 1$ still contain a factor of $(z - z_0)$ to some power, so they all evaluate to zero.

We can continue to take derivatives, noting that each successive derivative knocks out one more term in the polynomial, and then evaluating what's left at $z = z_0$ knocks out all the remaining terms except for one. That is

$$p''(z_0) = 2d_2 \tag{4}$$

$$p^{(3)}(z_0) = 3 \times 2 d_3 \tag{5}$$

$$\vdots = \vdots$$

$$p^{(k)}(z_0) = k! d_k \tag{6}$$

This leads to the *Taylor form* of a polynomial centered at $z = z_0$.

Taylor form

$$p(z)=\sum_{k=0}^{n}\frac{p^{(k)}(z_0)}{k!}(z-z_0)^k \tag{7}$$

Example 1. Express $p(z)=z^5+3z+4$ in Taylor form, centered at $z_0=2$. We have

$$\begin{aligned}
p(2) &= 42 = d_0 && (8)\\
p'(z) &= 5z^4+3 && (9)\\
p'(2) &= 83 = d_1 && (10)\\
p^{(2)}(z) &= 20z^3 && (11)\\
p^{(2)}(2) &= 160 = 2d_2 && (12)\\
d_2 &= 80 && (13)\\
p^{(3)}(z) &= 60z^2 && (14)\\
p^{(3)}(2) &= 240 = 6d_3 && (15)\\
d_3 &= 40 && (16)\\
p^{(4)}(z) &= 120z && (17)\\
p^{(4)}(2) &= 240 = 24d_4 && (18)\\
d_4 &= 10 && (19)\\
p^{(5)}(z) &= 120 && (20)\\
p^{(5)}(2) &= 120 = 120d_5 && (21)\\
d_5 &= 1 && (22)
\end{aligned}$$

Thus we have

$$p(z)=42+83(z-2)+80(z-2)^2+40(z-2)^3+10(z-2)^4+(z-2)^5 \tag{23}$$

Example 2. Express $q(z)=z^{10}$ in Taylor form, centered at $z_0=2$. This time, I'll just write out the derivatives evaluated at $z=2$ to save a bit of space, as the derivatives themselves are standard ones for the derivatives of a power. We have

$$q(2) = 1024 = d_0 \tag{24}$$
$$d_1 = q'(2) = 5120 \tag{25}$$
$$d_2 = \frac{1}{2!}q''(2) = 11520 \tag{26}$$
$$d_3 = \frac{1}{3!}q^{(3)}(2) = 15360 \tag{27}$$
$$d_4 = \frac{1}{4!}q^{(4)}(2) = 13440 \tag{28}$$
$$d_5 = \frac{1}{5!}q^{(5)}(2) = 8064 \tag{29}$$
$$d_6 = \frac{1}{6!}q^{(6)}(2) = 3360 \tag{30}$$
$$d_7 = \frac{1}{7!}q^{(7)}(2) = 960 \tag{31}$$
$$d_8 = \frac{1}{8!}q^{(8)}(2) = 180 \tag{32}$$
$$d_9 = \frac{1}{9!}q^{(9)}(2) = 20 \tag{33}$$
$$d_{10} = \frac{1}{10!}q^{(10)}(2) = 1 \tag{34}$$

We can then plug these values into 2 to get the new form of the polynomial. I won't bother writing it all out as it's just a matter of substituting in values. Still, you can imagine how difficult it would be to solve this by the brute force method mentioned above. We would have 11 simultaneous equations to solve.

Example 3. Express $r(z) = (z-1)(z-2)^3$ in Taylor form centered at $z_0 = 2$. Although this is a factored polynomial, we can apply the same algorithm.

$$r(2) = 0 = d_0 \tag{35}$$
$$r'(2) = 0 = d_1 \tag{36}$$
$$r''(2) = 0 = d_2 \tag{37}$$
$$d_3 = \frac{1}{3!}r^{(3)}(2) = 1 \tag{38}$$
$$d_4 = \frac{1}{4!}r^{(4)}(2) = 1 \tag{39}$$

Thus we have

$$r(z) = (z-2)^3 + (z-2)^4 \tag{40}$$

For reference, both forms expand into

$$r(z) = z^4 - 7z^3 + 18z^2 - 20z + 8 \tag{41}$$

so they are, in fact, the same polynomial.

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