TAYLOR SERIES FOR COMPLEX FUNCTIONS

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The Taylor series is first encountered as a theorem for real functions. In fact, it also applies to complex functions, the proof of which is given in Saff and Snider, Section 5.2. The Taylor series is defined as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots \quad (1)$$

$$= \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j \quad (2)$$

The theorem applies to the case where $f(z)$ is analytic within a disk $|z - z_0| < R$ for some radius R. Furthermore, the convergence is uniform on any closed disk $|z - z_0| \le R' < R$.

By replacing f by f' on the LHS of 2, we see that the Taylor series for f' can be obtained by differentiating the series for f term by term.

We'll give a few examples of Taylor series for common functions.

Example 1. For $f(z) = e^{-z}$ with $z_0 = 0$ we have

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= -e^{-0} = -1 \\ f''(0) &= e^{-0} = 1 \\ &\vdots \\ f^{(j)}(0) &= (-1)^j \end{aligned} \quad (3)$$

so

$$e^{-z} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^j \quad (4)$$

Since e^{-z} is entire, this series is valid over the entire complex plane.

Example 2. For $f(z) = \cosh z$ with $z_0 = 0$ we have

$$\begin{aligned} f(0) &= \cosh 0 = 1 \\ f'(0) &= \sinh 0 = 0 \\ &\vdots \\ f^{(j)}(0) & \begin{cases} 1 & j \text{ even} \\ 0 & j \text{ odd} \end{cases} \end{aligned} \tag{5}$$

so

$$\cosh z = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \tag{6}$$

Again, $\cosh z$ is entire, so the series is valid over the entire complex plane.

Example 3. For $f(z) = \sinh z$ with $z_0 = 0$ we can take the derivative of 6 since $\sinh z = d(\cosh z)/dz$. Thus

$$\sinh z = \sum_{j=0}^{\infty} \frac{2jz^{2j-1}}{(2j)!} \tag{7}$$

$$= \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!} \tag{8}$$

We've replaced $2j-1$ in 7 by $2j+1$, since the first term in 7 is 0.

Example 4. For $f(z) = \frac{1}{1-z}$ with $z_0 = i$ we have

$$\begin{aligned} f(i) &= \frac{1}{1-i} \\ f'(i) &= \frac{1}{(1-i)^2} \\ f''(i) &= \frac{2}{(1-i)^3} \\ &\vdots \\ f^{(j)}(i) &= \frac{j!}{(1-i)^{j+1}} \end{aligned} \tag{9}$$

so

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{j!}{(1-i)^{j+1}} (z-i)^j \tag{10}$$

$$= \sum_{j=0}^{\infty} \frac{(z-i)^j}{(1-i)^{j+1}} \tag{11}$$

The series is valid over a disk centred at $z_0 = i$ up to the largest radius that doesn't include the point $z = 1$. The shortest distance from $z_0 = i$ to $z = 1$ is $\sqrt{2}$, so the series is valid for the disk $|z-i| < \sqrt{2}$.

Example 5. For $f(z) = \text{Log } (1-z)$ with $z_0 = 0$, we have

$$\begin{aligned}
f(0) &= 0 \\
f'(0) &= -\frac{1}{1-0} = -1 \\
f''(0) &= -\frac{1}{(1-0)^2} = -1 \\
f^{(3)}(0) &= -\frac{2}{(1-0)^3} = -2 \\
&\vdots \\
f^{(j)}(0) &= -\frac{(j-1)!}{(1-0)^j} = -(j-1)!
\end{aligned} \tag{12}$$

so

$$\text{Log } (1-z) = \sum_{j=1}^{\infty} \frac{-z^j}{j} \tag{13}$$

Since Log 0 is undefined, the series is valid over the disk $|z| < 1$.

Example 6. For $f(z) = z^3$ with $z_0 = 1$ we have

$$\begin{aligned}
f(1) &= 1 \\
f'(1) &= 3(1)^2 = 3 \\
f''(1) &= 6 \times 1 = 6 \\
f^{(3)}(1) &= 6
\end{aligned} \tag{14}$$

with all higher derivatives equal to zero, so

$$z^3 = 1+3(z-1)+\frac{6}{2!}(z-1)^2+\frac{6}{3!}(z-1)^3 \tag{15}$$

$$= 1+3(z-1)+3(z-1)^2+(z-1)^3 \tag{16}$$

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