TAYLOR SERIES

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Often it's useful to write a mathematical function as a *power series*. So what's a power series?

The idea is that if we have some function $f(x)$ we would like to be able to write it like this

$$f(x)=\sum_{n=0}^{\infty}a_nx^n \tag{1}$$

where the a_n are constants that need to be determined. One reason why it would be nice to be able to do this is that if x is quite small (less than 1 anyway), higher powers of x get smaller and smaller, so a decent approximation to the function for small x is a polynomial with only 2 or 3 terms in it.

At first glance, equation 1 doesn't look all that likely. But let's see if we can find values for the coefficients a_n anyway.

If we write out the sum explicitly and then take the first derivative of the function, we get

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots \tag{2}$$

$$f'(x)=0+a_1+2a_2x+3a_3x^2+\ldots \tag{3}$$

Now, if we set $x=0$, then the first line tells us that $a_0=f(0)$ and the second line says $a_1=f'(0)$. Maybe we're on to something here, so let's try another couple of derivatives and see where it gets us.

$$f''(x)=2a_2+(3)(2)a_3x+(4)(3)a_4x^2+(5)(4)a_5x^3+\ldots \tag{4}$$

$$f^{(3)}(x)=(3)(2)a_3+(4)(3)(2)a_4x+(5)(4)(3)a_5x^2\ldots \tag{5}$$

$$f^{(4)}(x)=(4)(3)(2)a_4+(5)(4)(3)(2)a_5x+\ldots \tag{6}$$

where the notation $f^{(n)}(x)$ means the n^{th} derivative of $f(x)$.

Setting $x=0$ in each of these lines gives us three more of the coefficients:

$$a_2 = \frac{1}{2} f''(0) \tag{7}$$

$$a_3 = \frac{1}{3 \times 2} f^{(3)}(0) \tag{8}$$

$$a_4 = \frac{1}{4 \times 3 \times 2} f^{(4)}(0) \tag{9}$$

I've written the numerical bits explicitly as products since I want the pattern to show up. What seems to be happening is that every time we take another derivative, we reduce the power of x in all terms by 1, and multiply each term in the sum by the exponent that just got reduced by 1. By the time a particular term is reduced to a constant (that is, by the time the power of x is reduced to 0), the numerical factor multiplying that term is $n!$ (n factorial $= n(n-1)(n-2)\ldots(2)(1)$) provided that the *original* power of x in that term was n (that is, the power of x in that term when you expanded the original function $f(x)$). So in general:

$$a_n = \frac{1}{n!} f^{(n)}(0) \tag{10}$$

and

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n \tag{11}$$

This expansion uses $x = 0$ as the reference point, but it is fairly easy to generalize the formula so that we can use any point $x = a$ as a reference. If we start with the series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{12}$$

and work through the same process as above by calculating successive derivatives, we can isolate expressions for the coefficients a_n by setting $x = a$ (rather than $x = 0$) at each step, and we get the more general formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n \tag{13}$$

Equation 13 is the general Taylor series expansion of a function about $x = a$; equation 11 is just the special case of this expansion about $x = 0$, but is known as a Maclaurin series.

Geometrically, the first term $a_0 = f(a)$ approximates $f(x)$ by a constant (horizontal line on the graph) that passes through the point $(a, f(a))$. The next term approximates $f(x)$ by its tangent line at $x = a$; that is, it provides the approximation $f(x) \approx a_0 + a_1(x-a) = f(a) + f'(a)(x-a)$. Including the next term uses a parabola to approximate the function and so on. It's remarkable that pretty well any sensible function can be approximated to any level of precision by a polynomial.

It seems plausible that the Taylor series converges if $|x-a| < 1$ since successive powers will get smaller as n increases. However, it's not quite so obvious that the series converges at all, let alone to $f(x)$, if $|x-a| > 1$. Clearly it will do so only if the coefficient $f^{(n)}(a)/n!$ gets smaller faster than $|x-a|^n$ gets larger.

In fact, in all non-pathological cases (ones where the function and all its derivatives exist and are continuous between $x = a$ and the point at which we wish to expand the function, which we'll call $x = b$) *Taylor's theorem* reassures us that the series does in fact converge.

More precisely, Taylor's theorem (which we won't prove here, since the proof is a bit long and doesn't really contribute to our understanding of how to use the theorem) states that if the function $f(x)$ and its first n derivatives exist and are continuous on the *closed* interval $[a, b]$ (a 'closed' interval includes its endpoints), and the $(n+1)^{th}$ derivative exists on the *open* interval (a, b) (not including the endpoints), then

$$f(b) = \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(a)(b-a)^j + R_n \tag{14}$$

where the *remainder term* is given by

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \tag{15}$$

where c is some number (usually unknown) between a and b.

The problem of showing the series converges thus reduces to showing that

$$\lim_{n\to\infty} R_n = 0 \tag{16}$$

As an example, consider the function $f(x) = e^x$ expanded about $x = 0$. Since all derivatives of e^x are also e^x we have $f^{(n)}(0) = e^0 = 1$ for all n, and the Taylor (or Maclaurin) series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{17}$$

Writing it in the notation of the theorem, we need to show that

$$\lim_{n\to\infty}\frac{(x-0)^{n+1}}{(n+1)!}=0 \tag{18}$$

This is the same limit as $\lim_{n\to\infty}x^n/n!$. Clearly this limit is 0 if $|x|<1$, but what if $|x|>1$? Since x is a fixed value in this limit (n is the only thing that varies), we can choose some constant integer $M>|x|$. Now when $n>M$, we can write $n!=1\times2\times3\times\ldots\times M\times(M+1)\times\ldots\times n=M!(M+1)\times\ldots\times n$. Now $|x|^n/n!=|x|^M|x|^{n-M}/(M!(M+1)\times\ldots\times n)$. So we can factor out the term $|x|^M/M!$ since it is a constant (independent of n) and consider the limit of what's left:

$$\frac{|x|^M}{M!}\lim_{n\to\infty}\frac{|x|^{n-M}}{(M+1)\times\ldots\times n}=\frac{|x|^M}{M!}\lim_{n\to\infty}\frac{|x|}{M+1}\frac{|x|}{M+2}\times\ldots\times\frac{|x|}{n} \tag{19}$$

Each factor on the right is less than 1 (since $M>|x|$) so the effect of multiplying the expression by an infinite number of factors, each less than 1, gives us the required limit of zero. Thus by Taylor's theorem, the remainder term for the expansion tends to zero, and the expansion for e^x converges for all x.

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