TRIANGLE INEQUALITIES

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The triangle inequality is a useful theorem, usually first encountered in elementary geometry, but also applicable to real and complex analysis. Given two complex numbers

$$\begin{aligned} z_1 &= a_1 + ia_2 \\ z_2 &= b_1 + ib_2 \\ z_1 + z_2 &= (a_1 + b_1) + i(a_2 + b_2) \end{aligned} \tag{1}$$

(where a_i and b_i are real numbers), the triangle inequality states that

$$\boxed{|z_1 + z_2| \le |z_1| + |z_2|} \tag{2}$$

There are several ways this can be proved. Probably the easiest is to regard z_1 and z_2 as vectors in the complex plane. In that scenario, $|z_1|$ and $|z_2|$ are the lengths of the vectors representing z_1 and z_2. The sum $z_1 + z_2$ forms the third side of a triangle with z_1 and z_2 being the other two sides. The length of any side of a triangle must be less than (or equal to, if the triangle compresses into a line segment) the sum of the other two sides. If it weren't, you couldn't form a triangle from those three line segments.

If you don't trust your geometric intuition, there is an alternative proof based entirely on algebra. Starting from 1 we have

$$0 \le (a_1 b_2 - a_2 b_1)^2 \tag{3}$$

This follows because a_i and b_i are real, and the square of any real number is non-negative. Following on, we multiply out the square to get

$$0 \le (a_1 b_2)^2 - 2(a_1 b_2)(a_2 b_1) + (a_2 b_1)^2 \tag{4}$$

$$2(a_1 b_2)(a_2 b_1) \le (a_1 b_2)^2 + (a_2 b_1)^2 \tag{5}$$

$$(a_1 b_1)^2 + 2a_1 b_2 a_2 b_1 + (a_2 b_2)^2 \le (a_1 b_1)^2 + (a_1 b_2)^2 + (a_2 b_1)^2 + (a_2 b_2)^2 \tag{6}$$

$$(a_1b_1+a_2b_2)^2 \le \left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right) \tag{7}$$

$$a_1b_1+a_2b_2 \le \left(a_1^2+a_2^2\right)^{1/2}\left(b_1^2+b_2^2\right)^{1/2} \tag{8}$$

$$2a_1b_1+2a_2b_2 \le 2\left(a_1^2+a_2^2\right)^{1/2}\left(b_1^2+b_2^2\right)^{1/2} \tag{9}$$

$$a_1^2+2a_1b_1+b_1^2+a_2^2+2a_2b_2+b_2^2 \le a_1^2+b_1^2+a_2^2+b_2^2+2\left(a_1^2+a_2^2\right)^{1/2}\left(b_1^2+b_2^2\right)^{1/2} \tag{10}$$

$$(a_1+b_1)^2+(a_2+b_2)^2 \le a_1^2+b_1^2+a_2^2+b_2^2+2\left(a_1^2+a_2^2\right)^{1/2}\left(b_1^2+b_2^2\right)^{1/2} \tag{11}$$

$$(a_1+b_1)^2+(a_2+b_2)^2 \le \left[\left(a_1^2+a_2^2\right)^{1/2}+\left(b_1^2+b_2^2\right)^{1/2}\right]^2 \tag{12}$$

$$\left[(a_1+b_1)^2+(a_2+b_2)^2\right]^{1/2} \le \left(a_1^2+a_2^2\right)^{1/2}+\left(b_1^2+b_2^2\right)^{1/2} \tag{13}$$

$$|z_1+z_2| \le |z_1|+|z_2| \tag{14}$$

We take the positive square root in every case where a quantity is raised to the power of $\frac{1}{2}$. The last line follows on comparison with 1.

Reverse triangle inequality

We can also derive the *reverse triangle inequality*, which compares 2 complex numbers to their difference. In the regular triangle inequality, we take $z_1 = w_1$ and $z_2 = w_2 - w_1$. Then we have

$$|z_1+z_2| = |w_1+w_2-w_1| = |w_2| \le |w_1|+|w_2-w_1| \tag{15}$$

Rearranging, we have

$$|w_2|-|w_1| \le |w_2-w_1| \tag{16}$$

We can swap w_1 and w_2 in the derivation to get

$$|w_1|-|w_2| \le |w_1-w_2| = |w_2-w_1| \tag{17}$$

In other words, since $|w_1|-|w_2| = -(|w_2|-|w_1|)$ both of the statements are true:

$$\begin{aligned} |w_2|-|w_1| &\le |w_2-w_1| \\ -(|w_2|-|w_1|) &\le |w_2-w_1| \end{aligned} \tag{18}$$

Thus we have that both the quantity $|w_2|-|w_1|$ and its negative are less than $|w_2-w_1|$. We can write this as the reverse triangle inequality:

$$\boxed{||w_2|-|w_1|| \le |w_2-w_1|} \tag{19}$$