TRIGONOMETRIC INTEGRALS EXTRA

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The technique of using contour integration to evaluate some trigonometric integrals can be used to find some truly horrendous-looking integrals. Here we present one such example.

Suppose we have

$$I=\int_{0}^{2\pi}e^{\cos\theta}\cos\left(n\theta-\sin\theta\right)d\theta \tag{1}$$

This looks suitably frightening, but let's see what we can do.

The aim is to convert this to a contour integral around the unit circle $C:|z|=1$. That is, we have

$$\begin{aligned} z &= e^{i\theta}=\cos\theta+i\sin\theta \\ dz &= ie^{i\theta}d\theta \\ d\theta &= \frac{dz}{iz} \end{aligned} \tag{2}$$

We can rewrite 1 as the real part of another integral:

$$I=\Re\int_{0}^{2\pi}e^{\cos\theta}e^{i(n\theta-\sin\theta)}d\theta \tag{3}$$

Rearranging, we have

$$\int_{0}^{2\pi}e^{\cos\theta}e^{i(n\theta-\sin\theta)}d\theta=\int_{0}^{2\pi}e^{\cos\theta-i\sin\theta}e^{in\theta}d\theta \tag{4}$$

We can convert this into a contour integral by using 2. We have

$$\cos\theta-i\sin\theta=\frac{1}{z} \tag{5}$$

$$e^{in\theta}=z^{n} \tag{6}$$

This gives

$$I = \Re \oint_C e^{1/z} z^n \frac{dz}{iz} \tag{7}$$

$$= \Re \left[(-i) \oint_C e^{1/z} z^{n-1} dz \right] \tag{8}$$

The integrand has an essential singularity due to the $1/z$ exponent at $z_0 = 0$, so we need the coefficient of z^{-1} in the Laurent series about $z_0 = 0$. Expanding the exponential we have

$$e^{1/z} z^{n-1} = z^{n-1} \left(1 + \frac{1}{z} + \ldots + \frac{1}{n! z^n} + \ldots \right) \tag{9}$$

$$= z^{n-1} + z^{n-2} + \ldots + \frac{1}{n! z} + \ldots \tag{10}$$

Thus the residue is

$$\text{Res}\,(z_0 = 0) = \frac{1}{n!} \tag{11}$$

The integral 1 thus becomes, using the residue theorem to evaluate 8:

$$I = \Re \left[2\pi i \,(-i)\, \frac{1}{n!} \right] = \frac{2\pi}{n!} \tag{12}$$

Our final result is

$$\boxed{\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}} \tag{13}$$

[This turns out to be an integral even Maple can't do!]

As a corollary to this result, we note that the integral 8 is actually real as it stands, so its imaginary part is zero. From 3 this gives us

$$\Im \left[\int_0^{2\pi} e^{\cos\theta} e^{i(n\theta - \sin\theta)} d\theta \right] = 0 \tag{14}$$

That is, the real integral is

$$\int_0^{2\pi} e^{\cos\theta} \sin(n\theta - \sin\theta)\, d\theta = 0 \tag{15}$$

We can see these results graphically for $n = 1$ in Figs 1 and 2. We can see that in Fig. 2, the two lobes do appear to cancel each other, giving an integral of zero.

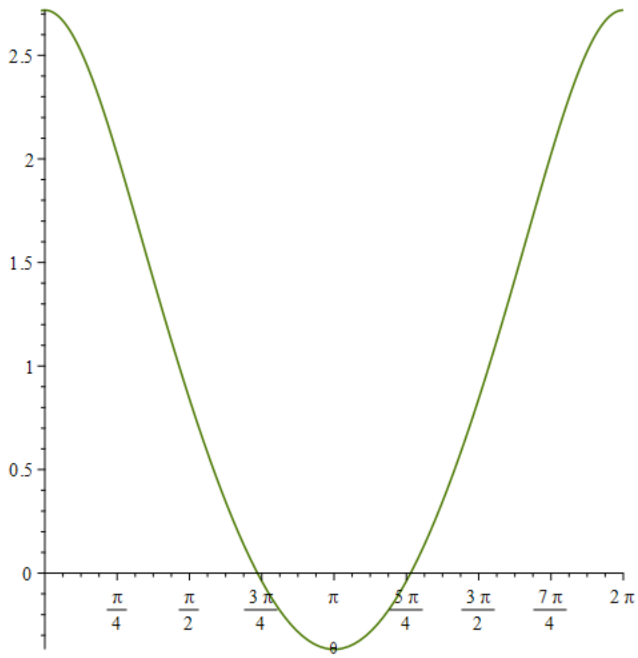

FIGURE 1. Plot of $e^{\cos\theta}\cos(\theta - \sin\theta)$.

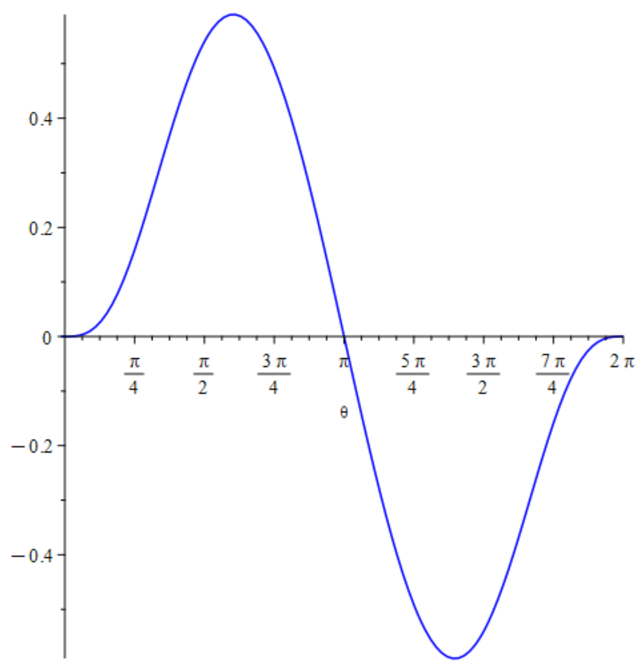

FIGURE 2. Plot of $e^{\cos\theta}\sin(\theta - \sin\theta)$.