TRIGONOMETRIC INTEGRALS

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We can use the Cauchy residue theorem to do some real trigonometric integrals over the ranges of integral multiples of π. The trick involves expressing the integrand in parametric form.

Suppose we have an integral of the form

$$\int_0^{2\pi} U(\cos\theta, \sin\theta)\, d\theta \tag{1}$$

We can express the trig functions in terms of complex exponentials in the usual way:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \tag{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \tag{3}$$

We can then express 1 as the parametrized form of a contour integral around the unit circle $|z|=1$, or $z=e^{i\theta}$. Along the contour we have

$$dz = ie^{i\theta}d\theta = izd\theta \tag{4}$$

so

$$d\theta = \frac{dz}{iz} \tag{5}$$

We can then rewrite the trig functions as

$$\begin{aligned} \cos\theta &= \frac{1}{2}\left(z+\frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}\left(z-\frac{1}{z}\right) \end{aligned} \tag{6}$$

and substitute into 1 to get a contour integral around the unit circle C. The limits of $\theta \in [0, 2\pi]$ correspond to a complete contour integral around the circle.

Example 1. Given

$$I=\int_0^{2\pi}\frac{d\theta}{2+\sin\theta} \tag{7}$$

Using 5 and 6 we have

$$I=\oint_C\frac{1}{2+\frac{1}{2i}\left(z-\frac{1}{z}\right)}\frac{dz}{iz} \tag{8}$$

$$=\oint_C\frac{1}{2iz+\frac{1}{2}\left(z^2-1\right)}dz \tag{9}$$

$$=\oint_C\frac{2}{z^2+4iz-1}dz \tag{10}$$

From the quadratic formula, we have

$$I=\oint_C\frac{2}{\left(z+\left(2+\sqrt{3}\right)i\right)\left(z+\left(2-\sqrt{3}\right)i\right)} \tag{11}$$

Thus there are simple poles at $\left(-2-\sqrt{3}\right)i$ and $\left(-2+\sqrt{3}\right)i$. Only the second pole lies within $|z|=1$. Using one of the techniques for calculating residues we have

$$\text{Res}\left(\left(-2+\sqrt{3}\right)i\right)=-\frac{\sqrt{3}}{3}i \tag{12}$$

so the integral is

$$\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}=2\pi i\left(-\frac{\sqrt{3}}{3}i\right) \tag{13}$$

$$=\frac{2\pi}{\sqrt{3}} \tag{14}$$

Example 2. Given

$$I=\int_0^{\pi}\frac{8}{5+2\cos\theta}d\theta \tag{15}$$

This time the integral is over $[0,\pi]$ rather than $[0,2\pi]$. However, we can use the sum of angles formula to note that

$$\cos\theta=\cos\left(2\pi-\theta\right) \tag{16}$$

Define

$$\phi = 2\pi - \theta \tag{17}$$
$$d\phi = -d\theta \tag{18}$$

so that

$$I = \int_{2\pi}^{\pi} \frac{8}{5+2\cos\phi}(-d\phi) \tag{19}$$
$$= \int_{\pi}^{2\pi} \frac{8}{5+2\cos\phi} d\phi \tag{20}$$
$$= \int_{\pi}^{2\pi} \frac{8}{5+2\cos\theta} d\theta \tag{21}$$

where in the last line, we just changed the dummy variable of integration. Therefore

$$\int_{0}^{2\pi} \frac{8}{5+2\cos\theta} d\theta = 2I \tag{22}$$

and we can use the contour method to solve this. We have

$$I = \int_{0}^{2\pi} \frac{4}{5+2\cos\theta} d\theta \tag{23}$$
$$= \oint_C \frac{4}{5+2\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz} \tag{24}$$
$$= \oint_C \frac{-4i}{z^2+5z+1} dz \tag{25}$$
$$= \oint_C \frac{-4i}{\left(z+\frac{5}{2}-\frac{\sqrt{21}}{2}\right)\left(z+\frac{5}{2}+\frac{\sqrt{21}}{2}\right)} dz \tag{26}$$

The only pole within $|z| = 1$ is at $z_0 = -\frac{5}{2}+\frac{\sqrt{21}}{2}$, where the residue is

$$\operatorname{Res}\left(-\frac{5}{2}+\frac{\sqrt{21}}{2}\right) = -\frac{4i}{\sqrt{21}} \tag{27}$$

so we have

$$\int_{0}^{\pi} \frac{8}{5+2\cos\theta} d\theta = 2\pi i\left(-\frac{4i}{\sqrt{21}}\right) \tag{28}$$
$$= \frac{8\pi}{\sqrt{21}} \tag{29}$$

Example 3. Given

$$I=\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^{2}\theta} \tag{30}$$

We can transform this to an integral over $[0,2\pi]$ by observing that

$$\sin(-\theta) = -\sin\theta \tag{31}$$
$$\sin(2\pi-\theta) = -\sin\theta \tag{32}$$

so

$$\sin^{2}(-\theta)=\sin^{2}\theta \tag{33}$$

Therefore, the integral over $[-\pi,0]$ is the same as the integral over $[\pi,2\pi]$ so

$$\begin{aligned} I &= \int_{0}^{2\pi}\frac{d\theta}{1+\sin^{2}\theta} && (34)\\ &= \oint_{C}\frac{1}{1+\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^{2}}\frac{dz}{iz} && (35)\\ &= \oint_{C}\frac{4iz}{z^{4}-6z^{2}+1}dz && (36)\end{aligned}$$

The polynomial in the denominator can be factored by using the quadratic formula twice, once to get z^2 and then to get z. The roots are

$$z_{0}=\sqrt{2}-1,-1-\sqrt{2},1-\sqrt{2},1+\sqrt{2} \tag{37}$$

Only $\sqrt{2}-1$ and $1-\sqrt{2}$ lie within $|z|=1$. The residues are

$$\begin{aligned}
\text{Res}\left(\sqrt{2}-1\right) &= \frac{4i\left(\sqrt{2}-1\right)}{2\left(\sqrt{2}-1\right)^2\left(-2+2\sqrt{2}\right)+12-12\sqrt{2}} \\
&= -\frac{\sqrt{2}}{4}i \\
\text{Res}\left(1-\sqrt{2}\right) &= \frac{4i\left(1-\sqrt{2}\right)}{2\left(1-\sqrt{2}\right)^2\left(2-2\sqrt{2}\right)-12+12\sqrt{2}} \\
&= -\frac{\sqrt{2}}{4}i
\end{aligned} \tag{38}$$

where Maple was used to do the calculations. We have

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta} = 2\pi i\left(-\frac{\sqrt{2}}{4}i-\frac{\sqrt{2}}{4}i\right) \tag{39}$$

$$= \sqrt{2}\pi \tag{40}$$

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