UNITARY OPERATORS - ACTIVE AND PASSIVE TRANSFORMATIONS OF AN OPERATOR

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A unitary operator transforms one orthonormal basis to another. Therefore if we have an operator Ω with matrix elements $\Omega_{ij}(\{f\}) = \langle f_i|\Omega|f_j\rangle$ in one orthonormal basis $(f_1,\ldots,f_n)$, we can transform the basis to another orthonormal basis $(e_1,\ldots,e_n)$ by a unitary transformation U so that

$$|e_i\rangle = U|f_i\rangle \tag{1}$$

This results in a transformation of the operator Ω's matrix elements:

$$\Omega_{ij}(\{e\}) = \langle e_i|\Omega|e_j\rangle \tag{2}$$

$$= \langle Uf_i|\Omega|Uf_j\rangle \tag{3}$$

$$= \left\langle f_i\left|U^\dagger \Omega U\right|f_j\right\rangle \tag{4}$$

Thus we can view the transformation as either a transformation of the basis vectors $e_i = Uf_i$, known as an *active transformation*, or as a transformation of the operator according to $\Omega \to U^\dagger \Omega U$, known as a *passive transformation*. The matrix elements of $U^\dagger \Omega U$ in the original basis $\{f\}$ are equal to the matrix elements of the original operator Ω in the new basis $\{e\}$.

We've already seen a few results about the trace and determinant of products of matrices. We'll list these here for reference:

- $\text{Tr}(\Omega\Lambda) = \text{Tr}(\Lambda\Omega)$. That is, even if the operators don't commute, the trace of a product of operators doesn't depend on the order of the operators in the product.
- The trace of a product of 3 or more operators is invariant under cyclic permutation. $\text{Tr}(\Omega\Lambda\theta) = \text{Tr}(\Lambda\theta\Omega) = \text{Tr}(\theta\Omega\Lambda)$. This follows directly from the previous result. For example, we can define $A \equiv \Lambda\theta$ so that $\text{Tr}(\Omega\Lambda\theta) = \text{Tr}(\Omega A) = \text{Tr}(A\Omega) = \text{Tr}(\Lambda\theta\Omega)$.
- The determinant of a unitary matrix is a complex number with modulus 1.

We can use these to prove a couple of further results about unitary transformations.

The trace of an operator is invariant under a unitary transformation:

$$\text{Tr}\left(U^{\dagger}\Omega U\right)=\text{Tr}\left(UU^{\dagger}\Omega\right)=\text{Tr}\Omega \tag{5}$$

since $UU^{\dagger}=I$.

Finally, the determinant of an operator is also invariant under a unitary transformation. Since the determinant of a product is the product of the determinants,

$$\det\left(U^{\dagger}\Omega U\right) = \det U^{\dagger}\det\Omega\det U \tag{6}$$

$$= e^{-i\alpha}\det\Omega e^{i\alpha} \tag{7}$$

$$= \det\Omega \tag{8}$$

In the second line we used the fact that $\det U$ is a complex number with unit modulus, and the fact that $\det U^{\dagger}=(\det U)^{*}$.

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