UNITARY OPERATORS

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Another important type of operator is the *unitary operator* U (Axler calls a unitary operator an *isometry*), which is defined by the condition that it is surjective and that

$$|Uu| = |u| \tag{1}$$

for all $u \in V$. That is, a unitary operator preserves the norm of all vectors. The identity matrix I is a special case of a unitary operator, as it doesn't change any vector, but multiplying I by any complex number α with $|\alpha| = 1$ also preserves the norm, so αI is another unitary operator.

Because U preserves the norm of all vectors, the only vector that can be in the null space of U is the zero vector, meaning that U is also injective. As it is both injective and surjective, it is invertible.

Theorem 1. *For a unitary operator U, $U^{\dagger} = U^{-1}$.*

Proof. From its definition and the properties of an adjoint operator, we have

$$\begin{aligned}
|Uu|^2 &= \langle Uu, Uu \rangle & (2)\\
&= \left\langle u, U^{\dagger}Uu \right\rangle & (3)\\
&= \langle u, u \rangle & (4)
\end{aligned}$$

Therefore, since U preserves the norm, we must have $U^{\dagger}Uu = u$, so $U^{\dagger}U = I$ so $U^{\dagger} = U^{-1}$. □

Theorem 2. *Unitary operators preserve inner products, meaning that $\langle Uu, Uv \rangle = \langle u, v \rangle$ for all $u, v \in V$.*

Proof. Since $U^{\dagger} = U^{-1}$ we have

$$\langle Uu, Uv \rangle = \left\langle u, U^{\dagger}Uv \right\rangle = \langle u, v \rangle \tag{5}$$

□

Theorem 3. *Acting on an orthonormal basis $(e_1, \ldots, e_n)$ with a unitary operator U produces another orthonormal basis.*

Proof. Suppose the orthonormal basis is converted to another set of vectors $(f_1, \ldots, f_n)$ by U:

$$f_i = Ue_i \tag{6}$$

Then

$$\langle f_i, f_j \rangle = \langle Ue_i, Ue_j \rangle = \langle e_i, e_j \rangle = \delta_{ij} \tag{7}$$

Thus $(f_1, \ldots, f_n)$ is an orthonormal set. Since the orthonormal basis $(e_1, \ldots, e_n)$ spans V (by assumption) and the set $(f_1, \ldots, f_n)$ contains n linearly independent orthonormal vectors, $(f_1, \ldots, f_n)$ is also an orthonormal basis for V. □

Theorem 4. *If one orthonormal basis $(e_1, \ldots, e_n)$ is converted to another $(f_1, \ldots, f_n)$ by a unitary operator U, then the matrix elements of U are the same in both bases.*

Proof. This is just a special case of the more general theorem that states that *any* operator that transforms one set of basis vectors into another has the same matrix elements in both bases. In this case, the proof is especially simple:

$$\begin{aligned}
U_{ki}(\{e\}) &= \langle e_k, Ue_i \rangle && (8)\\
&= \langle e_k, f_i \rangle && (9)\\
&= \left\langle U^{-1} f_k, f_i \right\rangle && (10)\\
&= \left\langle U^{\dagger} f_k, f_i \right\rangle && (11)\\
&= \langle f_k, Uf_i \rangle && (12)\\
&= U_{ki}(\{f\}) && (13)
\end{aligned}$$

□

References

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(4) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

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