UNITARY OPERATORS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 4 Jan 2021.

Another important type of operator is the *unitary operator* U (Axler calls a unitary operator an *isometry*), which is defined by the condition that it is surjective and that

$$|Uu| = |u| \tag{1}$$

for all $u \in V$. That is, a unitary operator preserves the norm of all vectors. The identity matrix I is a special case of a unitary operator, as it doesn't change any vector, but multiplying I by any complex number α with $|\alpha| = 1$ also preserves the norm, so αI is another unitary operator.

Because U preserves the norm of all vectors, the only vector that can be in the null space of U is the zero vector, meaning that U is also injective. As it is both injective and surjective, it is invertible.

Theorem 1. *For a unitary operator U, $U^\dagger = U^{-1}$.*

Proof. From its definition and the properties of an adjoint operator, we have

$$\begin{aligned} |Uu|^2 &= \langle Uu, Uu \rangle & (2) \\ &= \left\langle u, U^\dagger U u \right\rangle & (3) \\ &= \langle u, u \rangle & (4) \end{aligned}$$

Therefore, since U preserves the norm, we must have $U^\dagger U u = u$, so $U^\dagger U = I$ so $U^\dagger = U^{-1}$. □

Theorem 2. *Unitary operators preserve inner products, meaning that $\langle Uu, Uv \rangle = \langle u, v \rangle$ for all $u, v \in V$.*

Proof. Since $U^\dagger = U^{-1}$ we have

$$\langle Uu, Uv \rangle = \left\langle u, U^\dagger U v \right\rangle = \langle u, v \rangle \tag{5}$$

□

Theorem 3. *Acting on an orthonormal basis $(e_1, \ldots, e_n)$ with a unitary operator U produces another orthonormal basis.*

Proof. Suppose the orthonormal basis is converted to another set of vectors $(f_1,\ldots,f_n)$ by U:

$$f_i = Ue_i \tag{6}$$

Then

$$\langle f_i, f_j\rangle = \langle Ue_i, Ue_j\rangle = \langle e_i, e_j\rangle = \delta_{ij} \tag{7}$$

Thus $(f_1,\ldots,f_n)$ is an orthonormal set. Since the orthonormal basis $(e_1,\ldots,e_n)$ spans V (by assumption) and the set $(f_1,\ldots,f_n)$ contains n linearly independent orthonormal vectors, $(f_1,\ldots,f_n)$ is also an orthonormal basis for V. □

Theorem 4. *If one orthonormal basis $(e_1,\ldots,e_n)$ is converted to another $(f_1,\ldots,f_n)$ by a unitary operator U, then the matrix elements of U are the same in both bases.*

Proof. This is just a special case of the more general theorem that states that *any* operator that transforms one set of basis vectors into another has the same matrix elements in both bases. In this case, the proof is especially simple:

$$\begin{aligned}
U_{ki}(\{e\}) &= \langle e_k, Ue_i\rangle && (8)\\
&= \langle e_k, f_i\rangle && (9)\\
&= \left\langle U^{-1}f_k, f_i\right\rangle && (10)\\
&= \left\langle U^{\dagger}f_k, f_i\right\rangle && (11)\\
&= \langle f_k, Uf_i\rangle && (12)\\
&= U_{ki}(\{f\}) && (13)
\end{aligned}$$

□

References

(1) Axler, Sheldon (2015), *Linear Algebra Done Right*, 3rd edition, Springer. Chapter 7.
(2) Zwiebach, Barton, Online course *Mastering Quantum Mechanics Part 1: Wave Mechanics*. Archive available here.
(3) Landi, Giovanni & Zampini, Alessandro (2018), *Linear Algebra and Analytic Geometry for Physical Sciences*, Chapter 12.
(4) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 1.

PINGBACKS

Pingback: Normal operators
Pingback: Spectral theorem for normal operators
Pingback: Schrödinger equation and propagators
Pingback: Baker-Campbell-Hausdorff formula
Pingback: Time-dependent propagators
Pingback: Translational invariance in quantum mechanics
Pingback: Unitary operators - active and passive transformations of an operator
Pingback: Postulates of quantum mechanics - Schrödinger equation and propagators
Pingback: Time reversal, antiunitary operators and Wigner's theorem
Pingback: Unitary matrices - some examples
Pingback: Unitarity theorem for groups
Pingback: Time evolution operator