# UPPER LIMIT FOR SEQUENCES

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The *upper limit* of a sequence (not a series!) of real numbers $\{x_n\}_{n=1}^{\infty}$ is defined to be the smallest number $\ell$ with the property that for any $\varepsilon > 0$ there are only a finite number of values of $n$ such that $x_n$ exceeds $\ell + \varepsilon$. The upper limit is denoted by $\limsup x_n$. If there are no such numbers with that property, then we set $\limsup x_n = +\infty$, and if all real numbers have this property, then $\limsup x_n = -\infty$.

Note that the sequence itself need not have an limit in order to have an upper limit. For example, the sequence $x_n = (-1)^n$ doesn't have a limit since it alternates between $\pm 1$, but $\limsup x_n = +1$ since for any $\varepsilon > 0$ there are a finite number (actually zero) of $x_n$ that exceed $+1 + \varepsilon$.

For a sequence that *does* have a limit $x$, then $\limsup x_n = x$, since by the definition of a limit, there must be a finite $N$ such that for $n > N$, $x_n$ is closer to $x$ than the distance $\varepsilon$.

**Example 1.** Find $\limsup x_n$for

$$x_n = (-1)^n \frac{2n}{n+1} \tag{1}$$

Dividing top and bottom by $n$ we get

$$x_n = (-1)^n \frac{2}{1+1/n} \tag{2}$$

This tends to $(-1)^n 2$ as $n \to \infty$, and the fraction $\frac{2}{1+1/n}$ is always less than 2, so $\limsup x_n = 2$. Note again that this sequence doesn't have a limit due to the $(-1)^n$.

**Example 2.** Find $\limsup x_n$for

$$x_n = (-1)^n n \tag{3}$$

This sequence diverges to $\pm\infty$ so $\limsup x_n = +\infty$.

**Example 3.** Find $\limsup x_n$for

$$x_n = \frac{1}{n^2} \tag{4}$$

This sequence does have a limit, namely

$$\lim_{n\to\infty} x_n = 0 \tag{5}$$

so $\limsup x_n = 0$.

**Example 4.** Find $\limsup x_n$for

$$x_n = n\sin\left(\frac{n\pi}{2}\right) \tag{6}$$

The sine term cycles through the values $1,0,-1,0,\ldots$ Thus the values of $n$ when $\sin\left(\frac{n\pi}{2}\right) = 1$ are $1,5,9,\ldots$ and the sequence increases without limit so $\limsup x_n = +\infty$.

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**Cauchy-Hadamard formula.** It is shown in Saff and Snider, Section 5.4, that the radius of convergence $R$ of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{7}$$

is given by the Cauchy-Hadamard formula:

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} \tag{8}$$

We also know that, for convergent power series

$$R = \frac{1}{L} \tag{9}$$

where

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \tag{10}$$

For a convergent sequence of positive real numbers $\{x_n\}$

$$\limsup \frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \frac{1}{R} = \limsup \sqrt[n]{x_n} \tag{11}$$

In Saff and Snider's book, they claim that for *any* sequence of positive real numbers (even sequences that don't converge) we have

$$\limsup \sqrt[n]{x_n} \le \limsup \frac{x_{n+1}}{x_n} \tag{12}$$

It's true for some special cases, such as the divergent sequence $x_n = 2^n$, when we have

$$\begin{aligned} \limsup \sqrt[n]{2^n} &= 2 \\ \limsup \frac{2^{n+1}}{2^n} &= 2 \end{aligned} \tag{13}$$

I can't see how to prove this in general, so comments welcome.