VECTOR AREA

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The *vector area* of a surface is the integral of the differential area vector over the surface. That is

$$\mathbf{a}\equiv\int_S d\mathbf{a} \tag{1}$$

Remember that $d\mathbf{a}$ is normal to the surface at each point.

As an example, we can calculate $\mathbf{a}$ for a hemisphere of radius R. In spherical coordinates $|d\mathbf{a}|=R^2\sin\theta d\theta d\phi$. Since the vector area definition is a vector equation, it's easiest to split it up into 3 equations in rectangular coordinates. The normal to the hemisphere is in the $\hat{\mathbf{r}}$ direction at every point, and

$$\hat{\mathbf{r}}=\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}} \tag{2}$$

For the x component, we have

$$\begin{aligned} a_x &= R^2\int_0^{\pi/2}\int_0^{2\pi}\cos\phi\sin^3\theta d\phi d\theta && (3)\\ &= 0 && (4)\end{aligned}$$

since the integral over ϕ gives zero. Similarly, $a_y=0$. For a_z we get

$$\begin{aligned} a_z &= R^2\int_0^{\pi/2}\int_0^{2\pi}\cos\theta\sin\theta d\phi d\theta && (5)\\ &= \pi R^2 && (6)\end{aligned}$$

Thus the vector area of a hemisphere has the same magnitude as the circle that is its base. As a vector, the area is

$$\boldsymbol{a}=\pi R^2\hat{\boldsymbol{z}} \tag{7}$$

For a full sphere, we just extend the upper limit on θ to π in the above integrals, and we find that all three components are zero. In fact, this is a special case of a more general theorem, which is that $\mathbf{a}=0$ for *any* closed surface. To see this, we can apply the divergence theorem in the following way.

Suppose we have a vector field $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant vector and T is some scalar field. Then if S is a closed surface and V is the volume it encloses:

$$\int_S \mathbf{v}\cdot d\mathbf{a} = \int_V \nabla\cdot\mathbf{v}d^3\mathbf{r} \tag{8}$$

$$\mathbf{c}\cdot\int_S Td\mathbf{a} = \int_V \nabla\cdot(\mathbf{c}T)\,d^3\mathbf{r} \tag{9}$$

$$= \int_V [\mathbf{c}\cdot\nabla T + T\nabla\cdot\mathbf{c}]\,d^3\mathbf{r} \tag{10}$$

$$= \mathbf{c}\cdot\int_V \nabla T d^3\mathbf{r} \tag{11}$$

where the last equality follows because $\mathbf{c}$ is a constant so $\nabla\cdot\boldsymbol{c} = 0$. Since $\mathbf{c}$ is arbitrary, the two integrals must be equal:

$$\int_S Td\mathbf{a} = \int_V \nabla T d^3\mathbf{r} \tag{12}$$

If we take $T = 1$ then the LHS is just $\mathbf{a}$ and the RHS is zero since T is a constant. Thus the vector area of any closed surface is zero. This is because although the actual surface area is non-zero, the vector components always cancel each other out as we integrate over the surface.

Consider now a closed surface and divide it into two parts by cutting it along some closed curve L. Since the total vector area is zero, we must have $\mathbf{a}_{\text{upper}} = -\mathbf{a}_{\text{lower}}$. Now the shape of the surface is arbitrary, so for some curve L, we can keep the lower surface constant while varying the shape of the upper surface. That is, $\mathbf{a}_{\text{lower}}$ is held constant while the upper surface varies. However, the equality of the two areas must always hold, so *any* surface enclosed by L must have the same vector area. This explains why the vector area of a hemisphere is just the area of the circle that defines its base: these are just two different surfaces sharing a boundary curve.

We can apply this to get a different formula for $\mathbf{a}$ in terms of the boundary curve. Suppose we draw a cone with its vertex at the origin and with its base being the curve L. (The base need not lie in a plane, since L doesn't have to be flat. This won't affect the argument.) The vector area of this cone must be the same as any other surface that shares L. Now if we divide up L into line increments $d\mathbf{l}$ then we divide up the cone into a sequence of triangles with sides $\mathbf{r}$ (the vector from the vertex of the cone (the origin) to $d\mathbf{l}$), $d\mathbf{l}$ and $\mathbf{r}+d\mathbf{l}$. The area of this triangle is half the area of the parallelogram with two adjacent sides $\mathbf{r}$ and $d\mathbf{l}$, and that in turn is the magnitude of $\mathbf{r}\times d\mathbf{l}$. Therefore

$$\mathbf{a}=\frac{1}{2}\oint_{L}\mathbf{r}\times d\mathbf{l} \tag{13}$$

Finally, we can derive a similar result to 12 using Stokes's theorem. Again using a vector field $\mathbf{v}=\mathbf{c}T$ we have

$$\int_{L}\mathbf{v}\cdot d\mathbf{l}=\int_{S}\left[\nabla\times(\mathbf{c}T)\right]\cdot d\mathbf{a} \tag{14}$$

$$\mathbf{c}\cdot\int_{L}T d\mathbf{l}=\int_{S}\left[T\nabla\times\mathbf{c}-\mathbf{c}\times\nabla T\right]\cdot d\mathbf{a} \tag{15}$$

$$=-\int_{S}(\mathbf{c}\times\nabla T)\cdot d\mathbf{a} \tag{16}$$

$$=-\mathbf{c}\cdot\int_{S}(\nabla T\times d\mathbf{a}) \tag{17}$$

The last line uses the triple vector product identity $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$. Equating integrals gives

$$\int_{L}T d\mathbf{l}=-\int_{S}(\nabla T\times d\mathbf{a}) \tag{18}$$

If we let $T=\mathbf{c}\cdot\mathbf{r}$ for a constant vector $\mathbf{c}$, we get

$$\int_{L}(\mathbf{c}\cdot\mathbf{r})\,d\mathbf{l}=-\int_{S}\nabla(\mathbf{c}\cdot\mathbf{r})\times d\mathbf{a} \tag{19}$$

$$=-\int_{S}\left[\mathbf{c}\times(\nabla\times\mathbf{r})+(\mathbf{c}\cdot\nabla)\mathbf{r}\right]\times d\mathbf{a} \tag{20}$$

$$=-\int_{S}\mathbf{c}\times d\mathbf{a} \tag{21}$$

$$=-\mathbf{c}\times\mathbf{a}=\mathbf{a}\times\mathbf{c} \tag{22}$$

The second line omits the derivatives of $\mathbf{c}$ which are all zero. To get the third line, we use $\nabla\times\mathbf{r}=0$ and $(\mathbf{c}\cdot\nabla)\,\mathbf{r}=\mathbf{c}$ (both of which can be proved by direct calculation).

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