ZEROES AND SINGULARITIES

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A complex function f may be analytic over a domain, but be undefined at one or more isolated points. Such points are called *isolated singularities* and come in several forms. The details are given in Saff and Snider, section 5.6, so we'll summarize the essentials here.

Let f have an isolated singularity at z_0, and suppose that f has a Laurent series given by

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k \tag{1}$$

Then:

- If $a_k=0$ for all $k<0$, that is, the series has no terms with negative exponents, then f has a *removable singularity* at z_0, and its series is an ordinary Taylor series.
- If $a_{-m}\neq 0$ for some positive integer m, but $a_k=0$ for all $k<-m$, that is, the series has a finite but non-zero number of terms with negative exponents, then the function has a *pole* of order m at z_0. A pole of order 1 is a *simple pole*.
- If $a_k\neq 0$ for an infinite number of negative values of k, then f has an *essential singularity* at z_0.

There are several other ways of detecting singularities which are given in Saff and Snider, but the essential idea is that a removable singularity is one where the function can be redefined (or defined, if it wasn't originally) at z_0 to make it analytic, a pole is a singularity where the function blows up to $\pm\infty$, and an essential singularity which has rather bizarre properties which we outlined in a previous post.

A function has a *zero of order* m at z_0 if the lowest non-zero a_k is when $k=m$. In other words, the Taylor series about $z=z_0$ has no constant term and has the form

$$f(z)=\sum_{k=m}^{\infty}a_k(z-z_0)^k \tag{2}$$

From this form, we see that for a zero of order m, the function's first $m-1$ derivatives are zero at z_0.

Here are some examples of singularities of functions.

Example 1. $f(z) = \frac{z^3+1}{z^2(z+1)}$. The function as it stands is undefined at $z=0$ and $z=-1$. We can factor the numerator to get

$$f(z) = \frac{(z+1)\left(z^2-z+1\right)}{z^2(z+1)} \tag{3}$$

The $z+1$ factor can be cancelled, so f has a removable singularity at $z=-1$, and a pole of order 2 at $z=0$. The Laurent series about $z=0$ is (using Maple)

$$f(z) = \frac{1}{z^2} - \frac{1}{z} + 1 \tag{4}$$

This also shows the pole of order 2 at $z=0$.

The series about $z=-1$ is a Taylor series:

$$f(z) = 3 + 3(z+1) + 4(z+1)^2 + 5(z+1)^3 + 6(z+1)^4 + \ldots \tag{5}$$

which means this is a removable singularity.

Example 2. $f(z) = z^3 e^{1/z}$. We've shown before that $e^{1/z}$ has an essential singularity at $z=0$, so this function also has an essential singularity here. This also arises from the Laurent series which is

$$f(z) = \sum_{k=0}^{\infty} z^3 \frac{1}{k!z^k} \tag{6}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!z^{k-3}} \tag{7}$$

There are an infinite number of negative exponents in the series, so we have an essential singularity.

Example 3. $f(z) = \frac{\cos z}{z^2+1} + 4z$. The $\cos z$ is analytic over the entire plane, as is $4z$, so the only singularities are due to the z^2+1, which factors into $(z+i)(z-i)$. Thus we have simple poles at $z = \pm i$.

Example 4. $f(z)=\frac{1}{e^z-1}$.

The denominator is zero when $z=0$ and at $z=2k\pi i$ for integer k. Near $z=0$, we have

$$\frac{1}{e^z-1}=\frac{1}{1+z+\frac{z^2}{2!}+\ldots-1} \tag{8}$$

$$=\frac{1}{z+\frac{z^2}{2!}+\ldots} \tag{9}$$

$$=\frac{1}{z}\frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots} \tag{10}$$

For $|z|$ near zero, we can rewrite the reciprocal of the series as

$$\frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots}=1-\frac{z}{2!}+\mathcal{O}\left(z^2\right) \tag{11}$$

so we have

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\mathcal{O}(z) \tag{12}$$

Since the Laurent series contains z^{-1} as its only negative power, we have a simple pole at $z=0$.

For $z=2k\pi i$, we set

$$w=z-2k\pi i \tag{13}$$

so near the singularity, $|w|$ is very small. Also, we have

$$e^z=e^{w+2k\pi i}=e^w \tag{14}$$

so we can expand the exponential in terms of w instead of z, getting exactly the same series as above. Thus there are also simple poles at all $z_0=2k\pi i$.

Example 5. $f(z)=\tan z$. We have

$$\tan z=\frac{\sin z}{\cos z} \tag{15}$$

so the singularities are at the zeroes of $\cos z$ which are at $\left(n+\frac{1}{2}\right)\pi$. The zeroes are all simple order 1 zeroes since the derivative is $-\sin z$ which is nonzero at $\left(n+\frac{1}{2}\right)\pi$. Therefore $\tan z$ has simple poles at $z_0=\left(n+\frac{1}{2}\right)\pi$.

Example 6. $f(z) = \cos\left(1 - \frac{1}{z}\right)$. The definition of cosine

$$\cos w = \frac{e^{iw} + e^{-iw}}{2} \tag{16}$$

so here we have

$$\cos\left(1 - \frac{1}{z}\right) = \frac{1}{2}\left(e^{i}e^{-i/z} + e^{-i}e^{i/z}\right) \tag{17}$$

The factors $e^{\pm i}$ are well defined in terms of cos and sin. As we approach $z = 0$ along the positive or negative imaginary axis, $e^{-i/z}$ and $e^{i/z}$ behave like $e^{\pm 1/z}$. Because we know that $e^{1/z}$ has an essential singularity at $z = 0$, $\cos\left(1 - \frac{1}{z}\right)$ also has an essential singularity there.

Example 7. $f(z) = \frac{\sin(3z)}{z^3} - \frac{3}{z}$. We expand in series form:

$$\frac{\sin(3z)}{z^2} - \frac{3}{z} = \frac{1}{z^2}\left(3z - \frac{(3z)^3}{3!} + \frac{(3z)^5}{5!} + \ldots\right) - \frac{3}{z} \tag{18}$$

$$= -\frac{9}{2}z + \frac{81}{40}z^3 + \mathcal{O}\left(z^5\right) \tag{19}$$

This is a Taylor series, so we have a removable singularity at $z = 0$.

Example 8. $f(z) = \cot\frac{1}{z}$. We have

$$\cot\frac{1}{z} = \frac{\cos 1/z}{\sin 1/z} \tag{20}$$

so there are singularities at $z = 0$ and any values where $\sin 1/z = 0$. Because of the definition of $\sin w$ in terms of complex exponentials, there is an essential singularity at $z = 0$ (using the same argument as in Example 6 above). The other zeroes of $\sin 1/z$ occur at

$$\frac{1}{z} = k\pi \tag{21}$$

or, for $k \neq 0$

$$z = \frac{1}{k\pi} \tag{22}$$

To explore the behaviour near these points, we define

$$w = \frac{1}{z} - k\pi \tag{23}$$

Then

$$\sin\frac{1}{z} = \sin(w+k\pi) \tag{24}$$
$$= \sin w\cos k\pi + \sin k\pi\cos w \tag{25}$$
$$= (-1)^k \sin w \tag{26}$$

For z near $1/k\pi$, $|w|$ is very small, so we have

$$\sin w \approx w \tag{27}$$

and

$$\cot\frac{1}{z} = \frac{\cos w}{\sin w} \tag{28}$$
$$\approx \frac{(-1)^k}{w} \tag{29}$$
$$= \frac{(-1)^k}{\frac{1}{z} - k\pi} \tag{30}$$
$$= \frac{(-1)^{k+1}\, z/k\pi}{z - 1/k\pi} \tag{31}$$
$$= (-1)^{k+1}\left[\frac{1}{k^2\pi^2}\left(z - \frac{1}{k\pi}\right)^{-1} + \frac{1}{k\pi}\right] \tag{32}$$

where the last step can be verified by putting the two terms over a common denominator of $z - \frac{1}{k\pi}$. Thus $\cot\frac{1}{z}$ has a Laurent series with a single negative exponent, so it has simple poles at $z_0 = \frac{1}{k\pi}$ for $k \neq 0$, and an essential singularity at $z_0 = 0$.

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