FOUR-VELOCITY, MOMENTUM AND ENERGY

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In non-relativistic physics, the velocity of an object is a three dimensional vector whose components give the object's speed in each of three directions (the directions depend on the coordinate system). The velocity vector must be defined relative to a frame of reference.

Although the three-dimensional velocity can also be defined in relativity, there is a more general velocity that turns out to be more useful: the *four-velocity*. In general, an object's world line depends on its velocity and acceleration at each moment of time, all of which depend on the frame in which they are measured. In analogy with the non-relativistic velocity, which is defined as a vector tangent to an object's path in three-dimensional space, the relativistic four-velocity is defined as a four-vector that is tangent to an object's world line. Since four-vectors all transform using the Lorentz transformations, it is easiest to start with an object's four-velocity in its own rest frame, and then use the transformations to generate the four-velocity in other frames.

In its rest frame, an object's world line is just a line parallel to its time axis, so the four-velocity $\vec{U}$ is a vector whose only non-zero component is the time component U^0. That is

$$\vec{U} \xrightarrow{\mathcal{O}} (U^0, 0, 0, 0) \tag{1}$$

where $\mathcal{O}$ is the object's rest frame.

By convention, the vector's magnitude is taken to be -1:

$$\vec{U} \cdot \vec{U} = -1 \tag{2}$$

so in its rest frame,

$$\vec{U} \xrightarrow{\mathcal{O}} (1, 0, 0, 0) \tag{3}$$

How does this relate to the non-relativistic concept of velocity? To see this, we can apply the Lorentz transformations to see what the four-velocity looks like in frame $\bar{\mathcal{O}}$. We get

$$U^{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\alpha} U^{\alpha} \quad (4)$$

$$\vec{U} \overset{\bar{\mathcal{O}}}{\rightarrow} (\gamma, -v\gamma, 0, 0) \quad (5)$$

If $v \ll 1$ the x component of the four-velocity is

$$U^{\bar{1}} \approx -v \quad (6)$$

so it reduces to the non-relativistic velocity for speeds much lower than that of light. (Remember that frame $\bar{\mathcal{O}}$ is travelling at $+v$ relative to the object's rest frame, so the object's velocity in frame $\bar{\mathcal{O}}$ is $-v$.)

By analogy with classical physics, we can also define the *four-momentum* $\vec{p}$ by multiplying the four-velocity by the object's mass m.

$$\vec{p} \equiv m\vec{U} \quad (7)$$

The four-momentum thus transforms the same way as the four velocity since the only change is the multiplication by m. It also reduces to the classical momentum mv for small speeds. However, the time component of $\vec{p}$ seems out of place until we examine its limiting behaviour for small speeds

$$p^{\bar{0}} = \gamma m \quad (8)$$

$$= \frac{m}{\sqrt{1-v^2}} \quad (9)$$

$$\approx m + \frac{1}{2}mv^2 \quad (10)$$

where in the last line, we've used a Taylor expansion of the square root and retained only the first two terms.

The second term, $\frac{1}{2}mv^2$, is the classical kinetic energy of a particle. The first term is just the mass of the particle on its own, although had we been retaining factors of the speed of light c throughout these notes, the term would no doubt be more familiar, since it is mc^2.

As far as I can tell, at this stage Einstein essentially added another postulate to relativity when he proposed that the term $m/\sqrt{1-v^2}$ should be interpreted as the total energy of a particle moving at a speed v. Various arguments have been given to justify this conclusion but none of them (at least none that I have seen) qualify as a rigorous derivation of this result from the main two postulates of relativity: the invariance of inertial frames and the constancy of the speed of light. One reason why a logical jump is needed here is that if we are using the reduction to the classical form of the kinetic energy for small speeds as a justification, we need to remember that there are many other functions whose limiting form would give the same result, so that on its own is not enough to 'prove' that $m/\sqrt{1-v^2}$ is the particle's energy. Ultimately, the proposal relies on experimental evidence for its confirmation, and this particular aspect of relativity has been tested perhaps more than almost every other statement of the theory, since it is the basis of nuclear reactors and bombs, as well as the theory behind the particle reactions observed in accelerators.

The conclusion can thus be stated as

$$E=\frac{m}{\sqrt{1-v^2}} \tag{11}$$

where m is the *rest mass* of the particle. The four-momentum thus contains the ordinary vector momentum as its spatial components and the (scalar) energy as its time component.

$$\vec{p}=\left(E,p^1,p^2,p^3\right) \tag{12}$$

Note that this conclusion predicts that a particle's mass effectively increases as it speeds up, reaching infinity when $v=1$. This is another way of saying that a particle cannot reach the speed of light since an infinite force would be required to impart the infinite energy.

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