HYPERBOLIC ANGLES IN RELATIVITY

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We've seen in an earlier post that the invariance of the space-time interval in relativity means that we can use invariant hyperbolas to calibrate the axes of various observers. The use of hyperbolas, and hyperbolic functions, can be extended to provide something of an analogy to rotation in Euclidean space. The Lorentz transformation in one spatial and one time dimension is

$$L(v)=\gamma\left(\begin{array}{cc} 1 & v\\ v & 1 \end{array}\right) \tag{1}$$

where $\gamma=1/\sqrt{1-v^{2}}$. This matrix can also be written in the form

$$L(v)=\left(\begin{array}{cc} a & b\\ b & a \end{array}\right) \tag{2}$$

where $a=\gamma$ and $b=v\gamma$. These two quantities satisfy the identity

$$a^{2}-b^{2}=1 \tag{3}$$

so using the properties of the hyperbolic functions $\cosh u$ and $\sinh u$ we can set

$$\begin{aligned} a & \equiv \cosh u & (4)\\ b & \equiv \sinh u & (5) \end{aligned}$$

From these definitions, we have also

$$\begin{aligned} \tanh u & = \frac{\sinh u}{\cosh u} & (6)\\ & = v & (7)\\ u & = \tanh^{-1}v & (8) \end{aligned}$$

Using this definition, we can demonstrate a few properties of the Lorentz transformation using the hyperbolic identities. Our starting point is

$$L(u)=\begin{pmatrix}\cosh u & \sinh u\\ \sinh u & \cosh u\end{pmatrix} \tag{9}$$

where the relation between u and v is that given above: $u=\tanh^{-1}v$. This matrix bears a resemblance to the rotation matrix in 2-d Euclidean space:

$$R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{10}$$

This matrix, when applied to a 2-d Euclidean vector, rotates it by an angle θ about the origin. We will see that the Lorentz transformation does something similar to a spacetime vector, if we can define something akin to a hyperbolic angle.

First, if we apply two transformations in order

$$L(u)L(w)=\begin{pmatrix}\cosh u & \sinh u\\ \sinh u & \cosh u\end{pmatrix}\begin{pmatrix}\cosh w & \sinh w\\ \sinh w & \cosh w\end{pmatrix} \tag{11}$$

$$=\begin{pmatrix}\cosh u\cosh w+\sinh u\sinh w & \sinh u\cosh w+\cosh u\sinh w\\ \sinh u\cosh w+\cosh u\sinh w & \cosh u\cosh w+\sinh u\sinh w\end{pmatrix} \tag{12}$$

$$=\begin{pmatrix}\cosh(u+w) & \sinh(u+w)\\ \sinh(u+w) & \cosh(u+w)\end{pmatrix}=L(u+w) \tag{13}$$

That is, applying two separate transformations is equivalent to applying a single transformation with the sum of the arguments of the individual transformations.

In a similar way, we can show that

$$L^{-1}(u) = L(-u) \tag{14}$$
$$L(0) = I \tag{15}$$
$$L(u)L(w) = L(w)L(u) \tag{16}$$

The first equation states that the inverse transformation is the same as that using the negative argument. This is similar to the condition in a Euclidean rotation, where rotating by the negative angle $-\theta$ is the inverse of rotation by θ. The second condition says that if $u=0$ this amounts to the identity transformation in which nothing is changed. The third condition shows that the transformations are commutative, and follows directly from the addition property above.

These properties are all analogous to a rotation by an angle θ in 2-d Euclidean space, so we can think of the argument u of the hyperbolic functions as a kind of *hyperbolic angle*, and of the Lorentz transformation as a 'rotation' in this hyperbolic angle. Because $\tanh u$ tends to -1 as $u \to -\infty$ and to $+1$ as $u \to \infty$, the hyperbolic angle also tends to these infinite values the closer v gets to ± 1. Thus hyperbolic angles aren't periodic like Euclidean angles, so we can't push the analogy too far.

However, some other properties are common to both types of angle. Just as the length of a Euclidean vector doesn't change under rotation, so too is the interval between two events in spacetime invariant under a hyperbolic rotation. We showed in another post that the interval Δs^2 is invariant under the Lorentz transformation, and this follows somewhat more easily using hyperbolic functions. Indeed if

$$\left(\begin{array}{c} \Delta\bar{t} \\ \Delta\bar{x} \end{array}\right) = \left(\begin{array}{cc} \cosh u & \sinh u \\ \sinh u & \cosh u \end{array}\right)\left(\begin{array}{c} \Delta t \\ \Delta x \end{array}\right) \tag{17}$$

and if the interval in the first system is $\Delta x^2 - \Delta t^2 = \Delta s^2$ then by direct calculation, and the identity $\cosh^2 u - \sinh^2 u = 1$, it follows that $\Delta\bar{x}^2 - \Delta\bar{t}^2 = \Delta s^2$ as well.

In fact, the hyperbolic transformation preserves not only the spacetime interval, but also the distinction between past and future in the separation of events. Past and future are meaningful only for events where $\Delta s^2 \le 0$, since for spacelike separations with $\Delta s^2 > 0$, it is possible to find frames where the events are simultaneous ($\Delta t = 0$) so, given the continuity of the transformations between frames, there will also be some frames where the same pair of events can have $\Delta t > 0$ and $\Delta t < 0$ so we can't assign a definite order to events with a spacelike separation. Only for frames where it is impossible for Δt to be zero is it possible to define a unique order for a pair of events. In timelike frames where $\Delta t = t_2 - t_1 > 0$, event 2 always follows event 1, while with $\Delta t = t_2 - t_1 < 0$, event 2 always precedes event 1.

If we consider a pair of timelike events with $\Delta t < 0$, we must also have, since $\Delta s^2 = \Delta x^2 - \Delta t^2 < 0$, $-\Delta t < \Delta x < \Delta t$. From the transformation 17, we get

$$\Delta\bar{t} = \Delta t \cosh u + \Delta x \sinh u \tag{18}$$

We now have to consider the two cases $u > 0$ and $u < 0$ ($u = 0$ is just the identity transformation so nothing changes). If $u > 0$, then $\sinh u > 0$ and, since $\cosh u > 0$ for all u, we consider the case of the most positive value the second term can have and show that this still gives a negative value for

$\Delta\bar{t}$:

$$\begin{aligned}
\Delta\bar{t} &= \Delta t \cosh u + \Delta x \sinh u && (19)\\
&< \Delta t \cosh u - \Delta t \sinh u && (20)\\
&= \Delta t(\cosh u - \sinh u) && (21)\\
&= \Delta t\, e^{-u} && (22)\\
&< 0 && (23)
\end{aligned}$$

where we've used the definition of the hyperbolic functions in terms of exponentials, and the fact that the exponential function is always positive:

$$\begin{aligned}
\cosh u &= \frac{e^u + e^{-u}}{2} && (24)\\
\sinh u &= \frac{e^u - e^{-u}}{2} && (25)
\end{aligned}$$

If $u < 0$, then we again consider the most positive value of the second term to get

$$\begin{aligned}
\Delta\bar{t} &= \Delta t \cosh u + \Delta x \sinh u && (26)\\
&< \Delta t \cosh u + \Delta t \sinh u && (27)\\
&= \Delta t(\cosh u + \sinh u) && (28)\\
&= \Delta t\, e^{u} && (29)\\
&< 0 && (30)
\end{aligned}$$

So either way, the sign of Δt is preserved under the transformation. A similar calculation shows that intervals with $\Delta t > 0$ are also preserved. In that case, since $\Delta s^2 = \Delta x^2 - \Delta t^2 < 0$, $-\Delta t < \Delta x < \Delta t$ still holds, we must now choose the most negative value for the second term in each case. So for $u > 0$, we get

$$\begin{aligned}
\Delta\bar{t} &= \Delta t \cosh u + \Delta x \sinh u && (31)\\
&> \Delta t \cosh u - \Delta t \sinh u && (32)\\
&= \Delta t(\cosh u - \sinh u) && (33)\\
&= \Delta t\, e^{-u} && (34)\\
&> 0 && (35)
\end{aligned}$$

and for $u < 0$

$$\begin{aligned}
\Delta\bar{t} &= \Delta t\cosh u + \Delta x\sinh u && (36)\\
&> \Delta t\cosh u + \Delta t\sinh u && (37)\\
&= \Delta t(\cosh u + \sinh u) && (38)\\
&= \Delta t\, e^{u} && (39)\\
&> 0 && (40)
\end{aligned}$$

PINGBACKS

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