

## INVARIANT HYPERBOLAS

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We've seen that the space-time interval between two events, defined as

$$\Delta s^2 \equiv -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad (1)$$

is an invariant. That is, the interval between two events is the same as measured by all inertial observers. Returning to one spatial dimension, we can write the equation on which all pairs of events with a given interval lie:

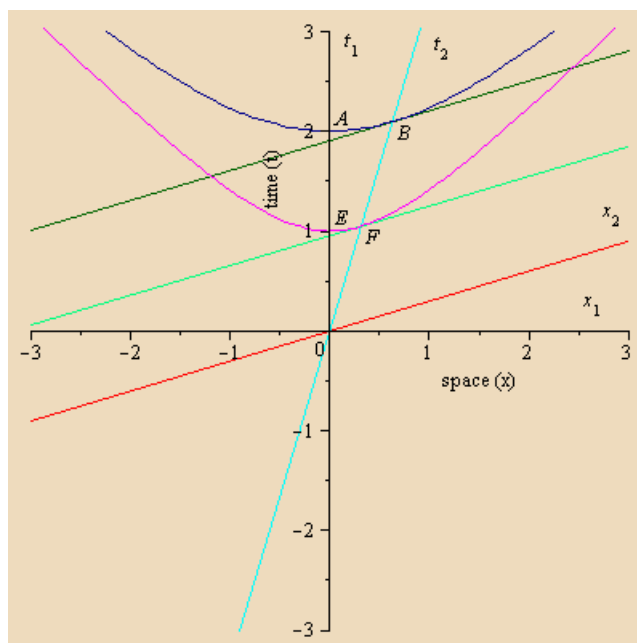
$$(\Delta x)^2 - (\Delta t)^2 = -a^2 \quad (2)$$

for some constant  $a$ . Since  $-a^2$  is always negative (we'll consider  $+a^2$  below), such intervals can have  $\Delta x = 0$  but not  $\Delta t = 0$ . That is, such events are always separated by a non-zero time interval, but it is possible for them to be observed to occur at the same place (although not all observers will see these two events in the same place).

If we take one of these events to be the origin, then we can write this equation in the form

$$x^2 - t^2 = -a^2 \quad (3)$$

which is the equation of a hyperbola. In the diagram, we've plotted the upper branch of two such hyperbolas, one (in magenta) with  $a = 1$  and the other (in dark blue) with  $a = 2$ . For the magenta hyperbola, the event  $E$  represents the point at which observer  $O_1$  sees  $x_1 = 0$ , and the point  $F$  is where  $O_2$  sees  $x_2 = 0$ . Thus the two observers disagree about events at which the two events occur at the same place. (There's nothing mysterious about this, since Galilean relativity predicts the same thing. A person in a moving train sees himself as at the same place in his frame, while an observer standing beside the track and watching the train go by sees the man moving.)



Similarly for the dark blue hyperbola. Here  $O_1$  sees event  $A$  occurring at  $x_1 = 0$  but  $O_2$  says it is event  $B$  that occurs at  $x_2 = 0$ .

Also, we can use these hyperbolas to calibrate one axis with respect to the other. In the diagram, if we use  $O_1$ 's axes as the starting point, we see that  $O_1$  measures event  $E$  to occur at  $x_1 = 0$ ,  $t_1 = 1$  while  $O_2$  measures event  $F$  to occur at  $x_2 = 0$ ,  $t_2 = 1$ . Thus as the hyperbola intersects the time axes of the various observers, it marks off the points at which each observer measures his local time variable to be 1. Similarly, the blue hyperbola marks off all the points where each observer measures his local time as 2.

An obvious question is whether these calibration points are linear. That is, if we choose the hyperbolas which intersect the  $t_1$  axis at values  $t_1 = 1$  and  $t_1 = a$ , is the ratio of the two times equal to  $a$  in all other frames as well? For example, in the figure, we know that the distance  $0E = EA = 1$ , but is it true that  $0F = FB$  as well? Clearly the distances  $0F$  and  $FB$  as measured on the space-time diagram are not equal to 1, but are they equal to each other so that the ratio of time measurements in  $O_2$ 's system is the same as that in  $O_1$ 's system?

We can verify this with a bit of algebra. We saw in an earlier post that the  $t_2$  axis has a slope of  $1/v$ , where  $v$  is the relative speed of the two systems. Therefore we can find the intersection point between the hyperbola 3 and the line with equation

$$t = \frac{1}{v}x \tag{4}$$

Substituting the line's equation into 3, we get

$$x^2 \left(1 - \frac{1}{v^2}\right) = -a^2 \quad (5)$$

$$x = \frac{av}{\sqrt{1-v^2}} \quad (6)$$

$$t = \frac{a}{\sqrt{1-v^2}} \quad (7)$$

This is the intersection point of the line and the hyperbola (e.g. the point  $B$  if  $a = 2$ , or  $F$  if  $a = 1$ ). The distance from the origin to this point is

$$d = \sqrt{x^2 + t^2} \quad (8)$$

$$= \sqrt{\frac{1+v^2}{1-v^2}} a \quad (9)$$

That is, the distance from the origin to the intersection point is directly proportional to  $a$ , so the ratios of time intervals in various frames are indeed the same.

Another feature of the invariant hyperbola that is useful is that the tangent line at the intersection point between the hyperbola and time axis is always parallel to the space axis. This can be seen in the diagram where the tangent to the lower hyperbola at point  $F$  is drawn in light green and can be seen to be parallel to the  $x_2$  axis, drawn in red. Similarly, the tangent to the upper hyperbola at point  $B$  (drawn in dark green) is also parallel to the  $x_2$  axis.

This can be shown algebraically. Using implicit differentiation with respect to  $x$  on 3, we get

$$2x - 2t \frac{dt}{dx} = 0 \quad (10)$$

$$\frac{dt}{dx} = \frac{x}{t} \quad (11)$$

We can now plug in the values of  $x$  and  $t$  at the intersection point (found above) to get

$$\frac{dt}{dx} \Big|_{t_2 \text{ axis}} = \frac{av}{\sqrt{1-v^2}} \frac{\sqrt{1-v^2}}{a} \quad (12)$$

$$= v \quad (13)$$

We saw in an earlier post that the slope of the  $x_2$  axis is indeed  $v$  so the tangent lines are parallel to the  $x_2$  axis.

Looking specifically at the magenta hyperbola (with  $a = 1$ ), we see that event  $F$  has time coordinate  $t_2 = 1$  in  $O_2$ 's system, but as measured in  $O_1$ 's system, the time as calculated above when we calculated the coordinates of the intersection point is  $t_1 = 1/\sqrt{1-v^2}$ . That is,  $O_1$  thinks more time has elapsed between the event at the origin and the event at  $F$  than  $O_2$  does, so  $O_1$  thinks  $O_2$ 's clock is running slowly. This is the time dilation effect, which we'll examine more fully in another post.

We can do a similar analysis with hyperbolas that have the form

$$x^2 - t^2 = +a^2 \tag{14}$$

These curves allow  $t = 0$  but not  $x = 0$ . Thus events separated by a positive interval can be seen by some observers to be simultaneous, but never at the same place. These hyperbolas can be used to calibrate the  $x$  axes of various observers, and we find that the ratios of distances are the same to all observers. Finally, the tangent line to the intersection of the hyperbola with the  $x$  axis is parallel to the  $t$  axis.

#### PINGBACKS

Pingback: [Time dilation](#)

Pingback: [Hyperbolic angles in relativity](#)