LORENTZ TRANSFORMATIONS

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Although space-time diagrams can be used to analyze simple problems in relativity, a general algebraic transformation between two inertial frames is easier for most situations. These are the Lorentz transformations between the coordinates of two observers O_1 and O_2, where O_2 moves with speed v along the x_1 axis of O_1.

We saw in an earlier post that coordinates perpendicular to the direction of motion are the same in both coordinate systems, so assuming that the transformation between the systems is linear (we show that this can be proved from more realistic assumptions in another post), the most general transformation is

$$t_2 = at_1 + bx_1 \tag{1}$$

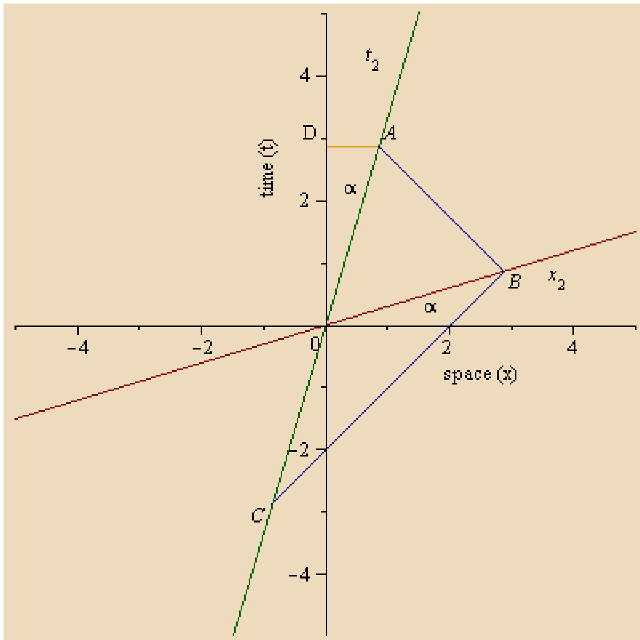
$$x_2 = dt_1 + ex_1 \tag{2}$$

$$y_2 = y_1 \tag{3}$$

$$z_2 = z_1 \tag{4}$$

where the coeffficients a, b, d and e are functions of v.

We can use the following space-time diagram to work out these coefficients. In the diagram, the main axes are those of O_1 with the axes of O_2 drawn in green and red.



We've seenthat the slope of the t_2 axis is $1/v$ and of the x_2 axis is v. So when $x_2 = 0$, we must have for the t_2 axis:

$$\begin{aligned} dt_1 + ex_1 &= 0 && (5)\\ t_1 &= \frac{1}{v}x_1 && (6)\\ &= -\frac{e}{d}x_1 && (7)\\ -v &= \frac{d}{e} && (8) \end{aligned}$$

Similarly, when $t_2 = 0$, we must have for the x_2 axis:

$$\begin{aligned} at_1 + bx_1 &= 0 && (9)\\ t_1 &= vx_1 && (10)\\ &= -\frac{b}{a}x_1 && (11)\\ -v &= \frac{b}{a} && (12) \end{aligned}$$

We can now write the transformations in terms of two coefficients.

$$\begin{aligned} t_2 &= a(t_1 - vx_1) && (13)\\ x_2 &= e(x_1 - vt_1) && (14) \end{aligned}$$

From the diagram, for the x_2 axis, if we pick $x_2 = 1$ we get from the second equation above:

$$1 = e(\cos\alpha - v\sin\alpha) \tag{15}$$

Similarly, for the t_2 axis, if we pick $t_2 = 1$ we get from the first equation above

$$1 = a(\cos\alpha - v\sin\alpha) \tag{16}$$

Comparing these two equations shows that

$$a = e \tag{17}$$

so we have

$$\begin{aligned} t_2 &= a(t_1 - vx_1) && (18)\\ x_2 &= a(x_1 - vt_1) && (19) \end{aligned}$$

For a pair of events, one of which is the origin, the invariance of the interval can be used to say

$$\begin{aligned} x_2^2 - t_2^2 &= x_1^2 - t_1^2 && (20)\\ &= a^2(x_1^2 - 2vx_1t_1 + v^2t_1^2 - t_1^2 + 2vt_1x_1 - v^2x_1^2) && (21)\\ &= a^2(1-v^2)(x_1^2 - t_1^2) && (22)\\ a &= \pm\frac{1}{\sqrt{1-v^2}} && (23) \end{aligned}$$

We must choose the + sign so that at $v = 0$ we get an identity.
This gives us the final form for the Lorentz transformations

$$\begin{aligned} t_2 &= \frac{1}{\sqrt{1-v^2}}(t_1 - vx_1) && (24)\\ x_2 &= \frac{1}{\sqrt{1-v^2}}(x_1 - vt_1) && (25)\\ y_2 &= y_1 && (26)\\ z_2 &= z_1 && (27) \end{aligned}$$

These transformations apply to the special case of inertial frames that are aligned so that O_2 travels with speed v along O_1's x_1 axis, with the corresponding y and z axes parallel in the two systems. This includes all real-life situations, although in some cases the coordinate systems may be more complex and the transformations take on a more involved form. However, all the standard relativistic effects such as time dilation and length contraction can be derived from these Lorentz transformations.

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