

## FOUR-VELOCITY: AN EXAMPLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 3; Problem P3.1.

We've already looked at the four-velocity in special relativity, but it's worth a second look from a different angle. We can instead define the four-velocity in terms of two events separated by an infinitesimal spacetime interval  $ds$ . The four-velocity is defined as the derivative of  $ds$  with respect to the proper time  $\tau$ , so that

$$(0.1) \quad u^i \equiv \frac{ds^i}{d\tau}$$

Since the components  $ds^i$  transform using the Lorentz transformation, then so do the components  $u^i$  of the four-velocity.

Since the spacetime interval is invariant (it has the same value in all inertial frames) the relation

$$(0.2) \quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

holds in all inertial frames. In particular, it holds in the observer's rest frame, in which  $dt = d\tau$ , so we have

$$(0.3) \quad ds^2 = -d\tau^2$$

In this rest frame, then, we get

$$(0.4) \quad u^i = \left( \frac{d\tau}{d\tau}, 0, 0, 0 \right)$$

$$(0.5) \quad = (1, 0, 0, 0)$$

The square of  $\mathbf{u}$  is then

$$(0.6) \quad \mathbf{u} \cdot \mathbf{u} = \eta_{ij} u^i u^j$$

$$(0.7) \quad = -1$$

where  $\eta_{ij}$  is the metric used in special relativity:

$$(0.8) \quad \eta_{ij} = \begin{cases} -1 & i = j = t \\ 1 & i = j = x, y, z \\ 0 & \text{otherwise} \end{cases}$$

Since this is true in the rest frame and the square of a four-vector is an invariant, it is true in all frames.

As an example, suppose we have an object that moves along a worldline given by (in some inertial frame)

$$(0.9) \quad x(\tau) = \frac{1}{g} [\cosh(g\tau) - 1]$$

( $y = z = 0$  and  $g$  is a constant). The  $x$  component of the four-velocity is then

$$(0.10) \quad u^x = \frac{dx}{d\tau}$$

$$(0.11) \quad = \sinh(g\tau)$$

Using  $\mathbf{u} \cdot \mathbf{u} = -1$  we can find the  $t$  component:

$$(0.12) \quad (u^x)^2 - (u^t)^2 = -1$$

$$(0.13) \quad \sinh^2(g\tau) + 1 = (u^t)^2$$

$$(0.14) \quad u^t = \cosh(g\tau)$$

using the identity  $\cosh^2 x - \sinh^2 x = 1$ . From this we can get the time in the inertial frame:

$$(0.15) \quad u^t = \frac{dt}{d\tau}$$

$$(0.16) \quad t(\tau) = \frac{1}{g} \sinh(g\tau)$$

The velocity of the object as seen in the inertial frame is

$$(0.17) \quad v_x = \frac{dx}{dt}$$

$$(0.18) \quad = \frac{u^x}{u^t}$$

$$(0.19) \quad = \tanh(g\tau)$$

Since  $\tanh$  is bounded by  $\pm 1$ , the velocity never exceeds 1, so never exceeds the speed of light.

We can invert the relation between proper time  $\tau$  and inertial time  $t$  to get

$$(0.20) \quad g\tau = \sinh^{-1}(gt)$$

Using the relations (derived from  $\cosh^2 x - \sinh^2 x = 1$ )

$$(0.21) \quad \sinh(\sinh^{-1} x) = x$$

$$(0.22) \quad \cosh(\sinh^{-1} x) = \sqrt{1+x^2}$$

$$(0.23) \quad \tanh(\sinh^{-1} x) = \frac{x}{\sqrt{1+x^2}}$$

we get

$$(0.24) \quad u^x(t) = gt$$

$$(0.25) \quad u^t(t) = \sqrt{1+(gt)^2}$$

$$(0.26) \quad v(t) = \frac{gt}{\sqrt{1+(gt)^2}}$$

Again, note that  $\mathbf{u} \cdot \mathbf{u} = -1$  and also that as  $t \rightarrow \infty$ ,  $v \rightarrow 1$  so again the velocity remains less than  $c$ .

#### PINGBACKS

Pingback: Four-acceleration

Pingback: Geodesic equation and four-velocity

Pingback: Schwarzschild metric: the Newtonian limit & Christoffel symbol worksheet