FOUR-VELOCITY: AN EXAMPLE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 3; Problem P3.1.

We've already looked at the four-velocity in special relativity, but it's worth a second look from a different angle. We can instead define the four-velocity in terms of two events separated by an infinitesimal spacetime interval $d\mathbf{s}$. The four-velocity is defined as the derivative of $d\mathbf{s}$ with respect to the proper time τ, so that

$$u^i \equiv \frac{ds^i}{d\tau} \tag{1}$$

Since the components ds^i transform using the Lorentz transformation, then so do the components u^i of the four-velocity.

Since the spacetime interval is invariant (it has the same value in all inertial frames) the relation

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

holds in all inertial frames. In particular, it holds in the observer's rest frame, in which $dt = d\tau$, so we have

$$ds^2 = -d\tau^2 \tag{3}$$

In this rest frame, then, we get

$$\begin{aligned} u^i &= \left(\frac{d\tau}{d\tau}, 0, 0, 0\right) && (4)\\ &= (1, 0, 0, 0) && (5) \end{aligned}$$

The square of $\mathbf{u}$ is then

$$\begin{aligned} \mathbf{u}\cdot\mathbf{u} &= \eta_{ij} u^i u^j && (6)\\ &= -1 && (7) \end{aligned}$$

where η_{ij} is the metric used in special relativity:

$$\eta_{ij}=\begin{cases}-1 & i=j=t\\ 1 & i=j=x,y,z\\ 0 & \text{otherwise}\end{cases} \tag{8}$$

Since this is true in the rest frame and the square of a four-vector is an invariant, it is true in all frames.

As an example, suppose we have an object that moves along a worldline given by (in some inertial frame)

$$x(\tau)=\frac{1}{g}\left[\cosh(g\tau)-1\right] \tag{9}$$

($y=z=0$ and g is a constant). The x component of the four-velocity is then

$$u^x = \frac{dx}{d\tau} \tag{10}$$

$$= \sinh(g\tau) \tag{11}$$

Using $\mathbf{u}\cdot\mathbf{u}=-1$ we can find the t component:

$$(u^x)^2-\left(u^t\right)^2 = -1 \tag{12}$$

$$\sinh^2(g\tau)+1 = \left(u^t\right)^2 \tag{13}$$

$$u^t = \cosh(g\tau) \tag{14}$$

using the identity $\cosh^2 x-\sinh^2 x=1$. From this we can get the time in the inertial frame:

$$u^t = \frac{dt}{d\tau} \tag{15}$$

$$t(\tau) = \frac{1}{g}\sinh(g\tau) \tag{16}$$

The velocity of the object as seen in the inertial frame is

$$v_x = \frac{dx}{dt} \tag{17}$$

$$= \frac{u^x}{u^t} \tag{18}$$

$$= \tanh(g\tau) \tag{19}$$

Since tanh is bounded by ± 1, the velocity never exceeds 1, so never exceeds the speed of light.

We can invert the relation between proper time τ and inertial time t to get

$$g\tau = \sinh^{-1}(gt) \tag{20}$$

Using the relations (derived from $\cosh^2 x - \sinh^2 x = 1$)

$$\sinh\left(\sinh^{-1} x\right) = x \tag{21}$$

$$\cosh\left(\sinh^{-1} x\right) = \sqrt{1+x^2} \tag{22}$$

$$\tanh\left(\sinh^{-1} x\right) = \frac{x}{\sqrt{1+x^2}} \tag{23}$$

we get

$$u^x(t) = gt \tag{24}$$

$$u^t(t) = \sqrt{1+(gt)^2} \tag{25}$$

$$v(t) = \frac{gt}{\sqrt{1+(gt)^2}} \tag{26}$$

Again, note that $\mathbf{u}\cdot\mathbf{u} = -1$ and also that as $t \to \infty$, $v \to 1$ so again the velocity remains less than c.

PINGBACKS

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Pingback: Geodesic equation and four-velocity

Pingback: Schwarzschild metric: the Newtonian limit & Christoffel symbol worksheet