

FOUR-VELOCITY: ANOTHER EXAMPLE

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 3; Problem P3.2.

Another example of four-velocity in special relativity. We start with an object whose velocity (3-d velocity, that is) in an inertial frame is

$$(0.1) \quad v_x(t) = \sqrt{1 - \frac{1}{(gt+1)^2}}$$

where g is a constant with relativistic units of m^{-1} , and $t \geq 0$ is the time measured in the inertial frame (so it's not the proper time of the object). The suffix x indicates the motion is along the x axis, as usual.

The relation between a proper time interval $d\tau$ and the time interval dt measured in a frame moving at speed v_x along the x axis with respect to the object is given by the time dilation formula

$$(0.2) \quad d\tau = dt \sqrt{1 - v_x^2}$$

This gives us the time component of the four-velocity:

$$(0.3) \quad u^t = \frac{dt}{d\tau}$$

$$(0.4) \quad = \frac{1}{\sqrt{1 - v_x^2}}$$

$$(0.5) \quad = 1 + gt$$

We can integrate this to get τ in terms of t :

$$(0.6) \quad \frac{dt}{1 + gt} = d\tau$$

$$(0.7) \quad \frac{1}{g} \ln(1 + gt) = \tau + \tau_0$$

where τ_0 is the constant of integration. If we require $\tau = 0$ when $t = 0$, then $\tau_0 = 0$ and we get

$$(0.8) \quad g\tau = \ln(1 + gt)$$

From this we get

$$(0.9) \quad u^t = e^{g\tau}$$

From the definition of four-velocity, we have

$$(0.10) \quad u^x \equiv \frac{dx}{d\tau}$$

$$(0.11) \quad = \frac{dx}{dt \sqrt{1 - v_x^2}}$$

$$(0.12) \quad = \frac{v_x}{\sqrt{1 - v_x^2}}$$

$$(0.13) \quad = (1 + gt) \sqrt{1 - \frac{1}{(gt + 1)^2}}$$

$$(0.14) \quad = \sqrt{(1 + gt)^2 - 1}$$

$$(0.15) \quad = \sqrt{e^{2g\tau} - 1}$$

We can now find x and t as functions of τ :

$$(0.16) \quad t(\tau) = \frac{1}{g}(e^{g\tau} - 1)$$

$$(0.17) \quad x(\tau) = \int_0^\tau \sqrt{e^{2g\tau'} - 1} d\tau'$$

$$(0.18) \quad = \frac{1}{g} \left[\sqrt{e^{2g\tau} - 1} - \arctan \left(\sqrt{e^{2g\tau} - 1} \right) \right]$$

using software to do the integral.

PINGBACKS

Pingback: Four-momentum conservation

Pingback: Particle orbits - conserved quantities

Pingback: Schwarzschild metric with negative mass