FOUR-MOMENTUM CONSERVATION: A TRIP TO ALPHA CENTAURI

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 3; Problem 3.6.

As a rather fanciful example of using the conservation of four-momentum, suppose we have a spaceship of total mass (including fuel) M, initially at rest on Earth. The fuel consists of matter/anti-matter which when mixed, produces photons that are ejected out of the back of the ship. If the ship burns enough fuel to accelerate to $v = 0.95$, then travels to some star system such as Alpha Centauri, then decelerates to zero for a landing, then, after some time at its destination it reverses the trip by again accelerating to $v = 0.95$, returning to Earth, and decelerating to rest, what is its final mass as a fraction of its initial mass?

We can assume that all motion takes place along the x axis, and treat the problem in the Earth's frame. Then the initial momentum is

$$\mathbf{p}_0 = [M, 0] \tag{1}$$

After accelerating, the combined momentum of the ship + ejected photons is

$$\mathbf{p}_1 = \gamma m_1 [1, 0.95] + E_1 [1, -1] \tag{2}$$

where $\gamma = 1/\sqrt{1-v^2} = 3.2$, E_1 is the energy of the ejected photons and m_1 is the mass of the ship after burning the fuel needed to accelerate.

By the conservation of momentum, we have $\mathbf{p}_1 = \mathbf{p}_0$ so

$$\gamma m_1 + E_1 = M \tag{3}$$

$$0.95\gamma m_1 - E_1 = 0 \tag{4}$$

Adding these 2 equations, we get

$$m_1 = \frac{M}{1.95\gamma} = 0.16M \tag{5}$$

Now to decelerate the ship, we eject the photons ahead of the ship, and we start with a momentum of $\gamma m_1 [1, 0.95]$. After deceleration, the mass is now m_2 and the ship is at rest, so we must have

$$\mathbf{p}_2 = [m_2, 0] + E_2 [1, 1] \tag{6}$$

Again, conservation of momentum requires $\mathbf{p}_2 = \gamma m_1 [1, 0.95]$, so

$$\begin{aligned} \gamma m_1 &= m_2 + E_2 && (7) \\ 0.95 \gamma m_1 &= E_2 && (8) \end{aligned}$$

Subtracting these equations we get

$$\begin{aligned} m_2 &= 0.05 \gamma m_1 && (9) \\ &= \frac{0.05}{1.95} M && (10) \end{aligned}$$

On the return trip, we go through exactly the same procedure, except we now start with a mass m_2 rather than M. Thus on the return to Earth, the ship's mass will be

$$\begin{aligned} m_E &= \frac{0.05}{1.95} m_2 && (11) \\ &= \left(\frac{0.05}{1.95} \right)^2 M && (12) \end{aligned}$$

Thus the ship's initial mass is

$$M = 1521 m_E \tag{13}$$

Virtually all the initial mass is fuel.

Incidentally, if we do this calculation for an arbitrary velocity v, we get

$$\begin{aligned} m_1 &= \frac{M}{(1+v)\gamma} && (14) \\ &= \sqrt{\frac{1-v}{1+v}} M && (15) \\ m_2 &= \frac{1-v}{1+v} M && (16) \\ m_E &= \left(\frac{1-v}{1+v} \right)^2 M && (17) \end{aligned}$$

Thus each acceleration or deceleration multiplies the previous mass by a factor of $\sqrt{\frac{1-v}{1+v}}$.