TENSOR INDEX NOTATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.1, 4.2.

Here are a few examples of the index notation and summation convention as used in tensor algebra. First, a summary of the rules for correct use of index notation:

(1) An index that is repeated twice within the same term, and where one instance is upper and the other lower, is to be summed over.
(2) A repeated index must not occur more than twice within a single term.
(3) Any index that is *not* repeated must occur in the same position (up or down) in all terms in an equation. (Exception: it's allowed to set a tensor expression to zero.)
(4) Any repeated (otherwise known as 'dummy' or 'bound') index may be renamed to any other symbol, provided it doesn't violate any of the other rules.
(5) Any single (otherwise known as 'free') index may be renamed to any other symbol, provided that symbol also occurs once only in each term.

Before proceeding, I should note that in relativity, some books use the convention that a Greek index can take on all values from 0 to 3 (that is, the time and all three space coordinates), while a Latin index takes on only the values 1,2,3 (space coordinates only). Other books reverse this convention. Sadly, the two books I've chosen to study (D'Inverno's and Moore's books) use opposite conventions. I'll stick with the Latin index for all 4 coordinates (which is what D'Inverno uses), since it's easier to type, but the reader should be aware that various books will use other conventions, so try to determine what the convention is in your favourite book before applying what you see here.

Some examples:

(1) $0 = m^2 + \left(p^i\right)^2$. This is invalid, since the m^2 term doesn't have any index, so it violates rule 3.
(2) $dF^{ij}/d\tau = 0$. This is OK, since it is an example of the exception to rule 3.
(3) $dp^i/d\tau = g$, where g is a constant. Invalid; violates rule 3.

(4) $F_{ab} = \eta_{ai}\eta_{bj}F^{ik}$. Invalid. Indexes j and k occur on RHS only; violates rule 3.
(5) $A^{ab} = \eta_{ai}\eta_{bj}F^{ij}$. Invalid, since a and b occur below on the RHS and above on the LHS.
(6) $A^i = \delta^i{}_a A^a$. OK.
(7) $0 = A^i + B^j$. Invalid, since the terms on the RHS have different free indexes.
(8) $qF^{ij} = \frac{dp^i}{d\tau}$. Invalid, since RHS has no index j.

Now some examples of renaming indexes:

(1) $A^2 = \eta_{ab}A^aA^b \implies A^2 = \eta_{ij}A^aB^b$. Wrong, since indexes on RHS are no longer repeated.
(2) $0 = \eta_{ab}A^b + \eta_{ai}B^i \implies 0 = \eta_{ab}\left(A^b + B^b\right)$. OK.
(3) $\eta_{ij} = \eta_{ab}\Lambda^a{}_i\Lambda^b{}_j \implies \eta_{ij} = \eta_{aa}\Lambda^a{}_i\Lambda^a{}_j$. Wrong, since a is repeated 4 times on RHS, so violates rule 2.
(4) $\frac{dp^i}{d\tau} = qF^{ij}\eta_{ja}u^a \implies \frac{dp^i}{d\tau} = qF^{ij}\eta_{ji}u^i$. Wrong, since i is a free index on LHS and is repeated 3 times on RHS.
(5) $\left(\Lambda^{-1}\right)^a{}_i\eta_{aj} = \eta_{ib}\Lambda^b{}_j \implies \left(\Lambda^{-1}\right)^b{}_i\eta_{bj} = \eta_{ia}\Lambda^a{}_j$. OK, as only dummy indexes have been relabelled and they still occur twice each after relabelling.