

METRIC TENSOR UNDER LORENTZ TRANSFORMATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.3, 4.4.

The invariant interval in special relativity can be written as

$$ds^2 = \eta_{ij} dx^i dx^j \quad (1)$$

where η_{ij} is the metric tensor in flat space, with components $\eta_{00} = -1$, $\eta_{ii} = +1$ for $i = 1, 2, 3$ and zero otherwise. Thus this relation is the same as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (2)$$

Under a Lorentz transformation, we get

$$ds^2 = \eta_{ij} dx'^i dx'^j \quad (3)$$

$$= \eta_{ij} \Lambda^i_a \Lambda^j_b dx^a dx^b \quad (4)$$

Since the interval is invariant, we get

$$\eta_{ij} \Lambda^i_a \Lambda^j_b dx^a dx^b = \eta_{ab} dx^a dx^b \quad (5)$$

$$\left(\eta_{ij} \Lambda^i_a \Lambda^j_b - \eta_{ab} \right) dx^a dx^b = 0 \quad (6)$$

Since the last equation must be true for an infinitesimal interval, the quantity in parentheses must be zero, so

$$\eta_{ab} = \eta_{ij} \Lambda^i_a \Lambda^j_b \quad (7)$$

That is, if we apply a Lorentz transformation (the *same* transformation!) to each index in the metric tensor, we get the same tensor back again.

We can multiply this equation by an inverse transformation to get

$$(\Lambda^{-1})^a_k \eta_{ab} = \eta_{ij} \Lambda^i_a \Lambda^j_b (\Lambda^{-1})^a_k \quad (8)$$

Multiplying a transformation by its inverse gives the identity matrix:

$$\Lambda^i_a (\Lambda^{-1})^a_k = \delta^i_k \quad (9)$$

So we get

$$(\Lambda^{-1})^a_k \eta_{ab} = \eta_{ij} \delta^i_k \Lambda^j_b \quad (10)$$

$$= \eta_{kj} \Lambda^j_b \quad (11)$$

Repeating the process, we get

$$(\Lambda^{-1})^b_l (\Lambda^{-1})^a_k \eta_{ab} = \eta_{kj} \Lambda^j_b (\Lambda^{-1})^b_l \quad (12)$$

$$= \eta_{kj} \delta^j_l \quad (13)$$

$$= \eta_{kl} \quad (14)$$

Thus, not surprisingly, if we multiply the metric tensor by two inverse Lorentz transformations, we get the same tensor back.

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