

METRIC TENSOR UNDER LORENTZ TRANSFORMATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.3, 4.4.

The invariant interval in special relativity can be written as

$$(0.1) \quad ds^2 = \eta_{ij} dx^i dx^j$$

where η_{ij} is the metric tensor in flat space, with components $\eta_{00} = -1$, $\eta_{ii} = +1$ for $i = 1, 2, 3$ and zero otherwise. Thus this relation is the same as

$$(0.2) \quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Under a Lorentz transformation, we get

$$(0.3) \quad ds^2 = \eta_{ij} dx'^i dx'^j$$

$$(0.4) \quad = \eta_{ij} \Lambda^i_a \Lambda^j_b dx^a dx^b$$

Since the interval is invariant, we get

$$(0.5) \quad \eta_{ij} \Lambda^i_a \Lambda^j_b dx^a dx^b = \eta_{ab} dx^a dx^b$$

$$(0.6) \quad \left(\eta_{ij} \Lambda^i_a \Lambda^j_b - \eta_{ab} \right) dx^a dx^b = 0$$

Since the last equation must be true for an infinitesimal interval, the quantity in parentheses must be zero, so

$$(0.7) \quad \eta_{ab} = \eta_{ij} \Lambda^i_a \Lambda^j_b$$

That is, if we apply a Lorentz transformation (the *same* transformation!) to each index in the metric tensor, we get the same tensor back again.

We can multiply this equation by an inverse transformation to get

$$(0.8) \quad (\Lambda^{-1})^a_k \eta_{ab} = \eta_{ij} \Lambda^i_a \Lambda^j_b (\Lambda^{-1})^a_k$$

Multiplying a transformation by its inverse gives the identity matrix:

$$(0.9) \quad \Lambda^i_a (\Lambda^{-1})^a_k = \delta^i_k$$

So we get

$$(0.10) \quad (\Lambda^{-1})^a_k \eta_{ab} = \eta_{ij} \delta^i_k \Lambda^j_b$$

$$(0.11) \quad = \eta_{kj} \Lambda^j_b$$

Repeating the process, we get

$$(0.12) \quad (\Lambda^{-1})^b_l (\Lambda^{-1})^a_k \eta_{ab} = \eta_{kj} \Lambda^j_b (\Lambda^{-1})^b_l$$

$$(0.13) \quad = \eta_{kj} \delta^j_l$$

$$(0.14) \quad = \eta_{kl}$$

Thus, not surprisingly, if we multiply the metric tensor by two inverse Lorentz transformations, we get the same tensor back.