METRIC TENSOR UNDER LORENTZ TRANSFORMATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.3, 4.4.

The invariant interval in special relativity can be written as

$$ds^2 = \eta_{ij} dx^i dx^j \tag{1}$$

where η_{ij} is the metric tensor in flat space, with components $\eta_{00} = -1$, $\eta_{ii} = +1$ for $i = 1,2,3$ and zero otherwise. Thus this relation is the same as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

Under a Lorentz transformation, we get

$$\begin{aligned} ds^2 &= \eta_{ij} dx'^i dx'^j & (3)\\ &= \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b dx^a dx^b & (4) \end{aligned}$$

Since the interval is invariant, we get

$$\begin{aligned} \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b dx^a dx^b &= \eta_{ab} dx^a dx^b & (5)\\ \left(\eta_{ij} \Lambda^i{}_a \Lambda^j{}_b - \eta_{ab}\right) dx^a dx^b &= 0 & (6) \end{aligned}$$

Since the last equation must be true for an infintesimal interval, the quantity in parentheses must be zero, so

$$\eta_{ab} = \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b \tag{7}$$

That is, if we apply a Lorentz transformation (the *same* transformation!) to each index in the metric tensor, we get the same tensor back again.

We can multiply this equation by an inverse transformation to get

$$\left(\Lambda^{-1}\right)^a{}_k \eta_{ab} = \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b \left(\Lambda^{-1}\right)^a{}_k \tag{8}$$

Multiplying a transformation by its inverse gives the identity matrix:

$$\Lambda^{i}{}_{a}\left(\Lambda^{-1}\right)^{a}{}_{k}=\delta^{i}{}_{k} \tag{9}$$

So we get

$$\begin{aligned}
\left(\Lambda^{-1}\right)^{a}{}_{k}\eta_{ab} &= \eta_{ij}\delta^{i}{}_{k}\Lambda^{j}{}_{b} && (10)\\
&= \eta_{kj}\Lambda^{j}{}_{b} && (11)
\end{aligned}$$

Repeating the process, we get

$$\begin{aligned}
\left(\Lambda^{-1}\right)^{b}{}_{l}\left(\Lambda^{-1}\right)^{a}{}_{k}\eta_{ab} &= \eta_{kj}\Lambda^{j}{}_{b}\left(\Lambda^{-1}\right)^{b}{}_{l} && (12)\\
&= \eta_{kj}\delta^{j}{}_{l} && (13)\\
&= \eta_{kl} && (14)
\end{aligned}$$

Thus, not surprisingly, if we multiply the metric tensor by two inverse Lorentz transformations, we get the same tensor back.

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