

## ELECTROMAGNETIC FIELD TENSOR: CONSERVATION OF MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.6.

As we'll study in more detail a bit later, the electric and magnetic fields can be combined into a single tensor known as the *electromagnetic field tensor*  $F^{ij}$ :

$$(0.1) \quad F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

We can see from its definition that this tensor is anti-symmetric, that is, that  $F^{ij} = -F^{ji}$ . For any anti-symmetric tensor we can show that

$$(0.2) \quad F^{ij}\eta_{ia}\eta_{jb}u^a u^b = 0$$

In this equation,  $\eta_{ij}$  is the metric tensor in flat space and  $u^a$  is the four-velocity, but in fact the formula is valid for any tensors  $\eta$  and  $u$ , provided that  $F$  is anti-symmetric. The proof involves a bit of index-switching.

$$(0.3) \quad F^{ij}\eta_{ia}\eta_{jb}u^a u^b = -F^{ji}\eta_{ia}\eta_{jb}u^a u^b$$

$$(0.4) \quad = -F^{ij}\eta_{ja}\eta_{ib}u^a u^b$$

$$(0.5) \quad = -F^{ij}\eta_{jb}\eta_{ia}u^b u^a$$

In the second line, we swapped the dummy indexes  $i$  and  $j$ , and in the third line we swapped  $a$  and  $b$ . The result shows that the original quantity is equal to its negative, which means it must be zero.

In terms of  $F^{ij}$ , the electric and magnetic (Lorentz) force laws for a charge  $q$  can be combined into a single equation:

$$(0.6) \quad \frac{dp^i}{d\tau} = qF^{ij}\eta_{ja}u^a$$

where  $u^a = \gamma[1, v_x, v_y, v_z]$  is the four-velocity.

For example, if  $i = 1$  we get

$$(0.7) \quad \frac{dp^1}{d\tau} = q\gamma(E_x + v_y B_z - v_z B_y)$$

In the non-relativistic limit,  $\gamma \rightarrow 1$  and this is the  $x$  component of the force law  $\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$ . We'll explore some of the other properties of this tensor later.

Since the square of the four-momentum of a particle is the negative of its mass squared ( $\mathbf{p} \cdot \mathbf{p} = \gamma^2 m^2 (-1 + v^2) = -m^2$ ), this should be conserved for a charged particle moving in an electromagnetic field. (Its *total* momentum is, of course, not conserved since the fields exert a force on the particle.)

We have

$$(0.8) \quad \frac{d(\mathbf{p} \cdot \mathbf{p})}{d\tau} = \frac{d}{d\tau} (\eta_{ij} p^i p^j)$$

$$(0.9) \quad = \eta_{ij} \left[ \frac{dp^i}{d\tau} p^j + p^i \frac{dp^j}{d\tau} \right]$$

$$(0.10) \quad = 2\eta_{ij} \frac{dp^i}{d\tau} p^j$$

$$(0.11) \quad = 2q\eta_{ij} F^{ik} \eta_{ka} u^a p^j$$

$$(0.12) \quad = 2qm F^{ik} \eta_{ij} \eta_{ka} u^a u^j$$

$$(0.13) \quad = 0$$

In the third line, we used the fact that  $\eta_{ij} = \eta_{ji}$  and swapped  $i$  and  $j$  in the second term. The fourth line uses 0.6 and the last line uses 0.2.

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