ELECTROMAGNETIC FIELD TENSOR: CONSERVATION OF MASS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.6.

As we'll study in more detail a bit later, the electric and magnetic fields can be combined into a single tensor known as the *electromagnetic field tensor* F^{ij}:

$$F^{ij}=\left[\begin{array}{cccc}
0 & E_{x} & E_{y} & E_{z}\\
-E_{x} & 0 & B_{z} & -B_{y}\\
-E_{y} & -B_{z} & 0 & B_{x}\\
-E_{z} & B_{y} & -B_{x} & 0
\end{array}\right] \tag{1}$$

We can see from its definition that this tensor is anti-symmetric, that is, that $F^{ij}=-F^{ji}$. For any anti-symmetric tensor we can show that

$$F^{ij}\eta_{ia}\eta_{jb}u^{a}u^{b}=0 \tag{2}$$

In this equation, η_{ij} is the metric tensor in flat space and u^{a} is the four-velocity, but in fact the formula is valid for any tensors η and u, provided that F is anti-symmetric. The proof involves a bit of index-switching.

$$\begin{aligned}
F^{ij}\eta_{ia}\eta_{jb}u^{a}u^{b} &= -F^{ji}\eta_{ia}\eta_{jb}u^{a}u^{b} && (3)\\
&= -F^{ij}\eta_{ja}\eta_{ib}u^{a}u^{b} && (4)\\
&= -F^{ij}\eta_{jb}\eta_{ia}u^{b}u^{a} && (5)
\end{aligned}$$

In the second line, we swapped the dummy indexes i and j, and in the third line we swapped a and b. The result shows that the original quantity is equal to its negative, which means it must be zero.

In terms of F^{ij}, the electric and magnetic (Lorentz) force laws for a charge q can be combined into a single equation:

$$\frac{dp^{i}}{d\tau}=qF^{ij}\eta_{ja}u^{a} \tag{6}$$

where $u^{a}=\gamma[1,v_{x},v_{y},v_{z}]$ is the four-velocity.

For example, if $i=1$ we get

$$\frac{dp^1}{d\tau} = q\gamma\left(E_x + v_y B_z - v_z B_y\right) \tag{7}$$

In the non-relativistic limit, $\gamma \to 1$ and this is the x component of the force law $\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$. We'll explore some of the other properties of this tensor later.

Since the square of the four-momentum of a particle is the negative of its mass squared ($\mathbf{p}\cdot\mathbf{p} = \gamma^2 m^2\left(-1+v^2\right) = -m^2$), this should be conserved for a charged particle moving in an electromagnetic field. (Its *total* momentum is, of course, not conserved since the fields exert a force on the particle.)

We have

$$\begin{aligned}
\frac{d\left(\mathbf{p}\cdot\mathbf{p}\right)}{d\tau} &= \frac{d}{d\tau}\left(\eta_{ij}p^i p^j\right) && (8)\\
&= \eta_{ij}\left[\frac{dp^i}{d\tau}p^j + p^i\frac{dp^j}{d\tau}\right] && (9)\\
&= 2\eta_{ij}\frac{dp^i}{d\tau}p^j && (10)\\
&= 2q\eta_{ij}F^{ik}\eta_{ka}u^a p^j && (11)\\
&= 2qmF^{ik}\eta_{ij}\eta_{ka}u^a u^j && (12)\\
&= 0 && (13)
\end{aligned}$$

In the third line, we used the fact that $\eta_{ij} = \eta_{ji}$ and swapped i and j in the second term. The fourth line uses 6 and the last line uses 2.

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