

## ELECTROMAGNETIC FIELD TENSOR: CHANGE IN KINETIC ENERGY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.7.

In the last post, we introduced the electromagnetic field tensor  $F^{ij}$ :

$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (1)$$

In terms of  $F^{ij}$ , the electric and magnetic (Lorentz) force laws for a charge  $q$  can be combined into a single equation:

$$\frac{dp^i}{d\tau} = qF^{ij}\eta_{ja}u^a \quad (2)$$

where  $u^a = \gamma[1, v_x, v_y, v_z]$  is the four-velocity. The three spatial components give the force law, but what about the time component? The time component of the four-momentum is the relativistic energy which, for a particle of rest mass  $m$  is  $\gamma m$ . If we expand this in a Taylor series for small  $v$ , we get

$$\gamma m = m + \frac{1}{2}mv^2 + \dots \quad (3)$$

Thus the relativistic energy is the rest mass plus the Newtonian kinetic energy (plus higher order terms). In the small- $v$  limit,  $\gamma \rightarrow 1$  and the  $t$  component of 2 therefore is

$$\frac{d}{d\tau} \left( m + \frac{1}{2}mv^2 \right) = q[\gamma E_x v_x + \gamma E_y v_y + \gamma E_z v_z] \quad (4)$$

$$= q\gamma \mathbf{v} \cdot \mathbf{E} \quad (5)$$

$$\approx q\mathbf{v} \cdot \mathbf{E} \quad (6)$$

Since the rest mass doesn't change, this equation is saying that the rate of change of kinetic energy is  $q\mathbf{v} \cdot \mathbf{E}$ . Does this make sense?

The force on a charge in an electric field is  $q\mathbf{E}$ , so the work done by this field in moving the charge through a distance  $d\mathbf{r}$  is  $q(d\mathbf{r} \cdot \mathbf{E})$ . This work

accelerates the charge, thus increasing its kinetic energy. The rate at which the kinetic energy increases is therefore  $q \frac{d}{dt} (d\mathbf{r} \cdot \mathbf{E}) = q\mathbf{v} \cdot \mathbf{E}$ .

Since the magnetic force on a charge is always perpendicular to the direction of motion, magnetic forces do no work, so there is no contribution to the kinetic energy from the magnetic field.