ELECTROMAGNETIC FIELD TENSOR: CHANGE IN KINETIC ENERGY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.7.

In the last post, we introduced the electromagnetic field tensor F^{ij}:

$$F^{ij}=\left[\begin{array}{cccc} 0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0 \end{array}\right] \tag{0.1}$$

In terms of F^{ij}, the electric and magnetic (Lorentz) force laws for a charge q can be combined into a single equation:

$$\frac{dp^i}{d\tau}=qF^{ij}\eta_{ja}u^a \tag{0.2}$$

where $u^a=\gamma[1,v_x,v_y,v_z]$ is the four-velocity. The three spatial components give the force law, but what about the time component? The time component of the four-momentum is the relativistic energy which, for a particle of rest mass m is γm. If we expand this in a Taylor series for small v, we get

$$\gamma m=m+\frac{1}{2}mv^2+\ldots \tag{0.3}$$

Thus the relativistic energy is the rest mass plus the Newtonian kinetic energy (plus higher order terms). In the small-v limit, $\gamma\to1$ and the t component of 0.2 therefore is

$$\begin{aligned} \frac{d}{d\tau}\left(m+\frac{1}{2}mv^2\right) &= q\left[\gamma E_xv_x+\gamma E_yv_y+\gamma E_zv_z\right] && (0.4)\\ &= q\gamma\mathbf{v}\cdot\mathbf{E} && (0.5)\\ &\approx q\mathbf{v}\cdot\mathbf{E} && (0.6) \end{aligned}$$

Since the rest mass doesn't change, this equation is saying that the rate of change of kinetic energy is $q\mathbf{v}\cdot\mathbf{E}$. Does this make sense?

The force on a charge in an electric field is $q\mathbf{E}$, so the work done by this field in moving the charge through a distance $d\mathbf{r}$ is $q(d\mathbf{r}\cdot\mathbf{E})$. This work accelerates the charge, thus increasing its kinetic energy. The rate at which the kinetic energy increases is therefore $q\frac{d}{dt}(d\mathbf{r}\cdot\mathbf{E}) = q\mathbf{v}\cdot\mathbf{E}$.

Since the magnetic force on a charge is always perpendicular to the direction of motion, magnetic forces do no work, so there is no contribution to the kinetic energy from the magnetic field.