ELECTROMAGNETIC FIELD TENSOR: CONTRACTIONS WITH METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.8, 4.9.

The electromagnetic field tensor F^{ij} is

$$F^{ij}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0\end{array}\right] \tag{1}$$

If we contract F^{ij} with the metric tensor in flat space, we get

$$\begin{aligned}\eta_{ij}F^{ij} &= -\eta_{ij}F^{ji} && (2)\\ &= -\eta_{ji}F^{ji} && (3)\\ &= -\eta_{ij}F^{ij} && (4)\end{aligned}$$

In the first line, we used $F^{ij}=-F^{ji}$; in the second, $\eta_{ij}=\eta_{ji}$ and in the last line, we swapped the dummy indexes i and j. Thus the original quantity is equal to its negative, so

$$\eta_{ij}F^{ij}=0 \tag{5}$$

Now consider $\eta_{ia}\eta_{jb}F^{ij}F^{ab}$. First, since $\eta_{00}=-1$ and $\eta_{ii}=+1$ for $i=1,2,3$:

$$\eta_{jb}F^{ij}=F^{i}{}_{b}=\left[\begin{array}{cccc}0 & E_x & E_y & E_z\\ E_x & 0 & B_z & -B_y\\ E_y & -B_z & 0 & B_x\\ E_z & B_y & -B_x & 0\end{array}\right] \tag{6}$$

This is because the only sign change occurs when $\eta_{jb}=-1$, which is when $j=b=0$, so all elements F^{i0} change sign.

Then

$$\eta_{ia}F^{i}{}_{b}=F_{ab}=\left[\begin{array}{cccc}0 & -E_{x} & -E_{y} & -E_{z}\\ E_{x} & 0 & B_{z} & -B_{y}\\ E_{y} & -B_{z} & 0 & B_{x}\\ E_{z} & B_{y} & -B_{x} & 0\end{array}\right] \tag{7}$$

This time, the sign change occurs when $i=a=0$, so elements F_{b}^{0} change sign. Thus lowering both indices in flat spacetime in rectangular coordinates changes the sign of all the electric field entries, but leaves the magnetic field unchanged.

Combining them, we get

$$\eta_{ia}\eta_{jb}F^{ij}F^{ab} = F_{ab}F^{ab}=-F_{ab}F^{ba} \tag{8}$$

To work this out, note that if we first formed the matrix product $F_{cb}F^{bd}$, then $F_{ab}F^{ba}$ is the sum of the diagonal elements of this matrix product. That is

$$\begin{aligned}-F_{ab}F^{ba} &= -\left(F_{0b}F^{b0}+F_{1b}F^{b1}+F_{2b}F^{b2}+F_{3b}F^{b3}\right) &(9)\\ &= -\left(E_{x}^{2}+E_{y}^{2}+E_{z}^{2}\right)-\left(E_{x}^{2}-B_{z}^{2}-B_{y}^{2}\right)-\left(E_{y}^{2}-B_{z}^{2}-B_{x}^{2}\right)-\left(E_{z}^{2}-B_{y}^{2}-B_{x}^{2}\right) &(10)\\ &= 2B^{2}-2E^{2} &(11)\end{aligned}$$

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