ELECTROMAGNETIC FIELD TENSOR: INVARIANCE UNDER LORENTZ TRANSFORMATIONS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.10.

The electromagnetic field tensor F^{ij} transforms between inertial frames by using 2 Lorentz transformations (we'll see why in a future post):

$$F'^{ij}=\Lambda^{i}{}_{a}\Lambda^{j}{}_{b}F^{ab} \tag{1}$$

Using this, let's see how the quantity $\eta_{ia}\eta_{jb}F^{ij}F^{ab}$ that we considered in the last post transforms. Since the metric tensor is invariant under Lorentz transformations

$$\eta_{ia}\eta_{jb}F'^{ij}F'^{ab}=\eta_{ia}\eta_{jb}\Lambda^{i}{}_{c}\Lambda^{j}{}_{d}F^{cd}\Lambda^{a}{}_{e}\Lambda^{b}{}_{f}F^{ef} \tag{2}$$

We can now use

$$\left(\Lambda^{-1}\right)^{a}{}_{k}\eta_{ab}=\eta_{kj}\Lambda^{j}{}_{b} \tag{3}$$

So we substitute in 2

$$\begin{aligned}
\eta_{ia}\Lambda^{i}{}_{c} &= \left(\Lambda^{-1}\right)^{i}{}_{a}\eta_{ic} && (4)\\
\eta_{jb}\Lambda^{j}{}_{d} &= \left(\Lambda^{-1}\right)^{j}{}_{b}\eta_{jd} && (5)
\end{aligned}$$

$$\begin{aligned}
&(6)\\
\eta_{ia}\eta_{jb}\Lambda^{i}{}_{c}\Lambda^{j}{}_{d}F^{cd}\Lambda^{a}{}_{e}\Lambda^{b}{}_{f}F^{ef} &= \left(\Lambda^{-1}\right)^{i}{}_{a}\eta_{ic}\left(\Lambda^{-1}\right)^{j}{}_{b}\eta_{jd}F^{cd}\Lambda^{a}{}_{e}\Lambda^{b}{}_{f}F^{ef}\\
&= \left(\Lambda^{-1}\right)^{i}{}_{a}\Lambda^{a}{}_{e}\eta_{ic}\left(\Lambda^{-1}\right)^{j}{}_{b}\Lambda^{b}{}_{f}\eta_{jd}F^{cd}F^{ef} && (7)\\
&= \delta^{i}{}_{e}\eta_{ic}\delta^{j}{}_{f}\eta_{jd}F^{cd}F^{ef} && (8)\\
&= \eta_{ec}\eta_{fd}F^{cd}F^{ef} && (9)
\end{aligned}$$

Thus the quantity $\eta_{ia}\eta_{jb}F^{ij}F^{ab}$ is invariant under Lorentz transformations. As we saw in the last post, this quantity is

$$\eta_{ia}\eta_{jb}F^{ij}F^{ab} = 2\left(B^2 - E^2\right) \tag{10}$$