

ELECTROMAGNETIC FIELD TENSOR: INVARIANCE UNDER LORENTZ TRANSFORMATIONS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.10.

The electromagnetic field tensor F^{ij} transforms between inertial frames by using 2 Lorentz transformations (we'll see why in a future post):

$$(0.1) \quad F'^{ij} = \Lambda^i_a \Lambda^j_b F^{ab}$$

Using this, let's see how the quantity $\eta_{ia}\eta_{jb}F'^{ij}F^{ab}$ that we considered in the last post transforms. Since the metric tensor is invariant under Lorentz transformations

$$(0.2) \quad \eta_{ia}\eta_{jb}F'^{ij}F^{ab} = \eta_{ia}\eta_{jb}\Lambda^i_c \Lambda^j_d F^{cd} \Lambda^a_e \Lambda^b_f F^{ef}$$

We can now use

$$(0.3) \quad (\Lambda^{-1})^a_k \eta_{ab} = \eta_{kj} \Lambda^j_b$$

So we substitute in 0.2

$$(0.4) \quad \eta_{ia}\Lambda^i_c = (\Lambda^{-1})^i_a \eta_{ic}$$

$$(0.5) \quad \eta_{jb}\Lambda^j_d = (\Lambda^{-1})^j_b \eta_{jd}$$

$$(0.6) \quad \eta_{ia}\eta_{jb}\Lambda^i_c \Lambda^j_d F^{cd} \Lambda^a_e \Lambda^b_f F^{ef} = (\Lambda^{-1})^i_a \eta_{ic} (\Lambda^{-1})^j_b \eta_{jd} F^{cd} \Lambda^a_e \Lambda^b_f F^{ef}$$

$$(0.7) \quad = (\Lambda^{-1})^i_a \Lambda^a_e \eta_{ic} (\Lambda^{-1})^j_b \Lambda^b_f \eta_{jd} F^{cd} F^{ef}$$

$$(0.8) \quad = \delta^i_e \eta_{ic} \delta^j_f \eta_{jd} F^{cd} F^{ef}$$

$$(0.9) \quad = \eta_{ec} \eta_{fd} F^{cd} F^{ef}$$

Thus the quantity $\eta_{ia}\eta_{jb}F'^{ij}F^{ab}$ is invariant under Lorentz transformations. As we saw in the last post, this quantity is

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$$(0.10) \quad \eta_{ia}\eta_{jb}F^{ij}F^{ab} = 2(B^2 - E^2)$$