

## ELECTROMAGNETIC FIELD TENSOR: INVARIANCE UNDER LORENTZ TRANSFORMATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 4; Problems 4.10.

The electromagnetic field tensor  $F^{ij}$  transforms between inertial frames by using 2 Lorentz transformations (we'll see why in a future post):

$$F'^{ij} = \Lambda^i_a \Lambda^j_b F^{ab} \quad (1)$$

Using this, let's see how the quantity  $\eta_{ia}\eta_{jb}F^{ij}F^{ab}$  that we considered in the last post transforms. Since the metric tensor is invariant under Lorentz transformations

$$\eta_{ia}\eta_{jb}F'^{ij}F'^{ab} = \eta_{ia}\eta_{jb}\Lambda^i_c \Lambda^j_d F^{cd} \Lambda^a_e \Lambda^b_f F^{ef} \quad (2)$$

We can now use

$$(\Lambda^{-1})^a_k \eta_{ab} = \eta_{kj} \Lambda^j_b \quad (3)$$

So we substitute in 2

$$\eta_{ia}\Lambda^i_c = (\Lambda^{-1})^i_a \eta_{ic} \quad (4)$$

$$\eta_{jb}\Lambda^j_d = (\Lambda^{-1})^j_b \eta_{jd} \quad (5)$$

$$\eta_{ia}\eta_{jb}\Lambda^i_c \Lambda^j_d F^{cd} \Lambda^a_e \Lambda^b_f F^{ef} = (\Lambda^{-1})^i_a \eta_{ic} (\Lambda^{-1})^j_b \eta_{jd} F^{cd} \Lambda^a_e \Lambda^b_f F^{ef} \quad (6)$$

$$= (\Lambda^{-1})^i_a \Lambda^a_e \eta_{ic} (\Lambda^{-1})^j_b \Lambda^b_f \eta_{jd} F^{cd} F^{ef} \quad (7)$$

$$= \delta^i_e \eta_{ic} \delta^j_f \eta_{jd} F^{cd} F^{ef} \quad (8)$$

$$= \eta_{ec} \eta_{fd} F^{cd} F^{ef} \quad (9)$$

Thus the quantity  $\eta_{ia}\eta_{jb}F^{ij}F^{ab}$  is invariant under Lorentz transformations. As we saw in the last post, this quantity is

$$\eta_{ia}\eta_{jb}F^{ij}F^{ab} = 2(B^2 - E^2) \quad (10)$$