METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 5; Problem 5.1.

Although the rectangular coordinate system is the most popular in many areas of physics, we frequently need to use other coordinate systems to make calculations easier. The most general types of coordinates are called *curvilinear coordinates*. We can define a coordinate system by first drawing in the curves (in 2-d) or surfaces (in 3-d or higher) where the various coordinates are constant. In rectangular coordinates in 2-d, horizontal lines correspond to $y=$ constant, and vertical lines to $x=$ constant. In polar coordinates, rays emanating from the origin correspond to $\theta=$ constant, and circles centred at the origin to $r=$ constant.

For each coordinate system, we can define a set of *basis vectors*. Again using 2-d as our working system, if we want to find the basis vectors at a particular point in the plane, we determine the coordinates of that point using the coordinate system we are currently dealing with, and then draw in the two coordinate curves through that point. For example, with polar coordinates if we chose the point (r_0,θ_0) we would draw in the circle $r=r_0$ and the ray $\theta=\theta_0$, both of which go through this point. We then specify a vector that is tangent to each of these coordinate curves at that point, and points in the direction of increase of the other coordinate. That is, the vector that is tangent to the curve $\theta=\theta_0$ points in the direction of increasing r, so it points radially outwards. The vector that is tangent to the circle $r=r_0$ points in the direction of increasing θ, so it points in a counterclockwise direction.

In higher dimensions, we select our point, then draw all the coordinate surfaces that intersect at that point. Using 3-d spherical coordinates, for example, we can choose a point (r_0,θ_0,ϕ_0). The surface $r=r_0$ is a sphere; the surface $\theta=\theta_0$ defines a cone centred at the origin, and $\phi=\phi_0$ defines a half-plane that starts at (and contains) the z axis. Then we choose three basis vectors, each of which is tangent to two of the surfaces and increases in the direction of the remaining coordinate. One such vector is tangent to the cone and the plane and points radially outward; this is the r basis vector. A second vector is tangent to the cone and the sphere and points horizontally, counter clockwise as viewed down the z axis; this is the ϕ

basis vector. Finally, the vector that is tangent to the sphere and the plane and points downward is the θ basis vector.

This recipe allows us to choose the *directions* of the basis vectors, but not their magnitudes. There are various ways the magnitudes can be defined. In the everyday use of polar, cylindrical and spherical coordinates, the basis vectors are defined as unit vectors. Using this convention results, in polar coordinates for example, in an infinitesimal displacement given as

$$d\mathbf{s} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} \tag{0.1}$$

Note the extra factor of r that is required due to the fact that an increment in angle of $d\theta$ causes a larger displacement the further away from the origin we are.

In relativity, it turns out that a different definition makes things easier. We define the basis vectors $\mathbf{e}_i$ to be such that the infinitesimal displacement between neighbouring points can be written as

$$d\mathbf{s} = dx^i\mathbf{e}_i \tag{0.2}$$

Comparing these two forms, we see that

$$\begin{aligned} \mathbf{e}_r &= \hat{\mathbf{r}} && (0.3)\\ \mathbf{e}_\theta &= r\hat{\theta} && (0.4) \end{aligned}$$

That is, the factor of r has been absorbed into the basis vector rather than left as an extra term that has to be added into the equations by hand, as it were. This, of course, means that in this system, $\mathbf{e}_\theta$ is *not* a unit vector.

In this system, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are still orthogonal, but in a more general coordinate system there is no need for basis vectors to be orthogonal; all that is really needed is that the basis vectors are linearly independent, so that they span the space. That is, in general we have a set of basis vectors $\mathbf{e}_i$ that span the space. In terms of this basis, the incremental distance squared is

$$\begin{aligned} ds^2 &= d\mathbf{s}\cdot d\mathbf{s} && (0.5)\\ &= \left(dx^i\mathbf{e}_i\right)\cdot\left(dx^j\mathbf{e}_j\right) && (0.6)\\ &= \mathbf{e}_i\cdot\mathbf{e}_j dx^i dx^j && (0.7)\\ &\equiv g_{ij}dx^i dx^j && (0.8) \end{aligned}$$

The quantity $g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j$ is the *metric tensor*. The fact that it is a tensor is a consequence of the invariance of ds^2 and the transformation properties of dx^i. We get

$$
\begin{aligned}
ds'^2 &= ds^2 && (0.9)\\
g'_{ij}dx'^i dx'^j &= g'_{ij}\frac{\partial x'^i}{\partial x^k}dx^k\frac{\partial x'^j}{\partial x^l}dx^l && (0.10)\\
&= g'_{ij}\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}dx^k dx^l && (0.11)\\
&= g_{kl}dx^k dx^l && (0.12)
\end{aligned}
$$

The first line results from the transformation of the dx^i and the last line results from the invariance of ds^2. Comparing the last two lines, we have

$$g_{kl}dx^k dx^l = g'_{ij}\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}dx^k dx^l \tag{0.13}$$

$$\left(g_{kl} - g'_{ij}\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}\right)dx^k dx^l = 0 \tag{0.14}$$

Since this relation must be true for any displacement, the quantity in parenthses must be zero, so

$$g_{kl} = g'_{ij}\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} \tag{0.15}$$

which shows that g_{ij} transforms like a covariant rank-2 tensor.

As an example, consider again 2-d polar coordinates. The transformation between polar and rectangular is

$$
\begin{aligned}
x &= r\cos\theta && (0.16)\\
y &= r\sin\theta && (0.17)\\
r &= \sqrt{x^2+y^2} && (0.18)\\
\theta &= \arctan\frac{y}{x} && (0.19)
\end{aligned}
$$

The partial derivatives are

$$\frac{\partial x}{\partial r} = \cos\theta \tag{0.20}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \tag{0.21}$$

$$\frac{\partial y}{\partial r} = \sin\theta \tag{0.22}$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta \tag{0.23}$$

The metric tensor in rectangular coordinates is

$$g_{xx} = g_{yy} = 1 \tag{0.24}$$

$$g_{xy} = g_{yx} = 0 \tag{0.25}$$

Using 0.15 we can find the metric in polar coordinates:

$$g_{rr} = g_{xx}\frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + g_{yy}\frac{\partial y}{\partial r}\frac{\partial y}{\partial r} \tag{0.26}$$

$$= \cos^2\theta + \sin^2\theta \tag{0.27}$$

$$= 1 \tag{0.28}$$

$$g_{\theta\theta} = g_{xx}\frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + g_{yy}\frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} \tag{0.29}$$

$$= (-r\sin\theta)^2 + (r\cos\theta) \tag{0.30}$$

$$= r^2 \tag{0.31}$$

Similarly, we can show that $g_{r\theta} = g_{\theta r} = 0$. The invariant interval is thus

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{0.32}$$

This is consistent with our definition of the basis vectors above.

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