VECTORS AND THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 5; Problems 5.2, 5.3.

We can define a general vector $\mathbf{A}$ in terms of the basis vectors $\mathbf{e}_i$ in a given coordinate system:

$$\mathbf{A} \equiv A^i \mathbf{e}_i \tag{1}$$

This is analogous to the definition of the infinitesimal displacement that we met earlier: $d\mathbf{s} = dx^i \mathbf{e}_i$. This has a couple of consequences. First, since the basis vectors are not necessarily either unit vectors or orthogonal, this definition may be different from the usual definition of a vector that you're used to from linear algebra courses.

Second, we require the transformation of a vector's components between coordinate systems to be the same as the components of $d\mathbf{s}$, which means that

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \tag{2}$$

Finally, the square of a vector follows the same pattern as the square of the increment ds^2:

$$A^2 = \mathbf{A} \cdot \mathbf{A} = g_{ij} A^i A^j \tag{3}$$

As an example, consider the case of uniform circular motion. From elementary physics, we know that, in polar coordinates, the radial component of the velocity $v^r = 0$ (since the object is always at the same distance from the origin) and the tangential component is $v^\theta = v$. Using the metric for polar coordinates, this means that

$$\begin{aligned}
v^2 &= g_{ij} v^i v^j && (4)\\
&= 1 \times 0 \times 0 + r^2 \times v^\theta \times v^\theta && (5)\\
&= \left(v^\theta\right)^2 r^2 && (6)\\
v^\theta &= \frac{v}{r} && (7)
\end{aligned}$$

To transform this vector to rectangular coordinates, we have

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j \tag{8}$$

where the primed system is rectangular and the unprimed is polar. So

$$\begin{aligned}
v^x &= \frac{\partial x}{\partial r} v^r + \frac{\partial x}{\partial \theta} v^\theta && (9)\\
&= -r\sin\theta \frac{v}{r} && (10)\\
&= -v\sin\theta && (11)\\
v^y &= \frac{\partial y}{\partial r} v^r + \frac{\partial y}{\partial \theta} v^\theta && (12)\\
&= r\cos\theta \frac{v}{r} && (13)\\
&= v\cos\theta && (14)
\end{aligned}$$

The square is invariant, since using the rectangular metric

$$\begin{aligned}
v^2 &= g_{ij} v^i v^j && (15)\\
&= (-v\sin\theta)^2 + (v\cos\theta)^2 && (16)\\
&= v^2 && (17)
\end{aligned}$$

Now let's look at the inverse problem. This time we have an object moving at a constant speed v in the $+y$ direction, so that $v^x = 0$, $v^y = v$. To convert this to polar coordinates, we need the derivatives

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} && (18)\\
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} && (19)\\
\frac{\partial \theta}{\partial x} &= -\frac{y}{x^2+y^2} && (20)\\
\frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} && (21)
\end{aligned}$$

Then we get

$$\begin{aligned} v^r &= \frac{y}{\sqrt{x^2+y^2}}v && (22)\\ &= v\sin\theta && (23)\\ v^\theta &= \frac{x}{x^2+y^2}v && (24)\\ &= \frac{\cos\theta}{r}v && (25)\end{aligned}$$

If the object starts at $(x,y)=(b,0)$ at $t=0$, then $y(t)=vt$ and $x(t)=b$. In polar coordinates we get

$$\begin{aligned} r(t) &= \sqrt{b^2+(vt)^2} && (26)\\ \theta(t) &= \arctan\frac{vt}{b} && (27)\\ v^r &= \frac{vt}{\sqrt{b^2+(vt)^2}}v && (28)\\ v^\theta &= \frac{b}{b^2+(vt)^2}v && (29)\end{aligned}$$

At $t=0$, the motion is entirely in the θ direction, since the object is moving tangent to the circle $r=b$ at that time. As time increases, the motion gradually transfers over to the radial direction, with $\lim_{t\to\infty}v^r=v$.

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