

VECTORS AND THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 5; Problems 5.2, 5.3.

We can define a general vector \mathbf{A} in terms of the basis vectors \mathbf{e}_i in a given coordinate system:

$$(0.1) \quad \mathbf{A} \equiv A^i \mathbf{e}_i$$

This is analogous to the definition of the infinitesimal displacement that we met earlier: $d\mathbf{s} = dx^i \mathbf{e}_i$. This has a couple of consequences. First, since the basis vectors are not necessarily either unit vectors or orthogonal, this definition may be different from the usual definition of a vector that you're used to from linear algebra courses.

Second, we require the transformation of a vector's components between coordinate systems to be the same as the components of $d\mathbf{s}$, which means that

$$(0.2) \quad A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

Finally, the square of a vector follows the same pattern as the square of the increment ds^2 :

$$(0.3) \quad A^2 = \mathbf{A} \cdot \mathbf{A} = g_{ij} A^i A^j$$

As an example, consider the case of uniform circular motion. From elementary physics, we know that, in polar coordinates, the radial component of the velocity $v^r = 0$ (since the object is always at the same distance from the origin) and the tangential component is $v^\theta = v$. Using the metric for polar coordinates, this means that

$$\begin{aligned}
 (0.4) \quad v^2 &= g_{ij}v^i v^j \\
 (0.5) \quad &= 1 \times 0 \times 0 + r^2 \times v^\theta \times v^\theta \\
 (0.6) \quad &= (v^\theta)^2 r^2 \\
 (0.7) \quad v^\theta &= \frac{v}{r}
 \end{aligned}$$

To transform this vector to rectangular coordinates, we have

$$(0.8) \quad v'^i = \frac{\partial x'^i}{\partial x^j} v^j$$

where the primed system is rectangular and the unprimed is polar. So

$$(0.9) \quad v^x = \frac{\partial x}{\partial r} v^r + \frac{\partial x}{\partial \theta} v^\theta$$

$$(0.10) \quad = -r \sin \theta \frac{v}{r}$$

$$(0.11) \quad = -v \sin \theta$$

$$(0.12) \quad v^y = \frac{\partial y}{\partial r} v^r + \frac{\partial y}{\partial \theta} v^\theta$$

$$(0.13) \quad = r \cos \theta \frac{v}{r}$$

$$(0.14) \quad = v \cos \theta$$

The square is invariant, since using the rectangular metric

$$(0.15) \quad v^2 = g_{ij}v^i v^j$$

$$(0.16) \quad = (-v \sin \theta)^2 + (v \cos \theta)^2$$

$$(0.17) \quad = v^2$$

Now let's look at the inverse problem. This time we have an object moving at a constant speed v in the $+y$ direction, so that $v^x = 0$, $v^y = v$. To convert this to polar coordinates, we need the derivatives

$$(0.18) \quad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$(0.19) \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$(0.20) \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}$$

$$(0.21) \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}$$

Then we get

$$(0.22) \quad v^r = \frac{y}{\sqrt{x^2 + y^2}} v$$

$$(0.23) \quad = v \sin \theta$$

$$(0.24) \quad v^\theta = \frac{x}{x^2 + y^2} v$$

$$(0.25) \quad = \frac{\cos \theta}{r} v$$

If the object starts at $(x, y) = (b, 0)$ at $t = 0$, then $y(t) = vt$ and $x(t) = b$. In polar coordinates we get

$$(0.26) \quad r(t) = \sqrt{b^2 + (vt)^2}$$

$$(0.27) \quad \theta(t) = \arctan \frac{vt}{b}$$

$$(0.28) \quad v^r = \frac{vt}{\sqrt{b^2 + (vt)^2}} v$$

$$(0.29) \quad v^\theta = \frac{b}{b^2 + (vt)^2} v$$

At $t = 0$, the motion is entirely in the θ direction, since the object is moving tangent to the circle $r = b$ at that time. As time increases, the motion gradually transfers over to the radial direction, with $\lim_{t \rightarrow \infty} v^r = v$.

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