

METRIC TENSOR: SINUSOIDAL COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 5; Problem 5.5.

As another example of a different coordinate system, we can define a sinusoidal system by introducing coordinates u and w defined as:

$$u = x \quad (1)$$

$$w = y - A \sin(bx) \quad (2)$$

where A and b are constants. If x and y have units of length, then b must have units of length^{-1} . If A has units of length, then u and w both have units of length.

The curves of constant u are just the same as the curves of constant x , that is, vertical lines. The curves of constant w are defined by $y = k + A \sin(bx)$. These are parallel horizontal sine curves displaced vertically by the constant k .

The inverted relations are

$$x = u \quad (3)$$

$$y = w + A \sin(bu) \quad (4)$$

We can work out the metric of the sinusoidal system from the rectangular metric by direct calculation

$$g'_{ij} = g_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} = \begin{bmatrix} 1 + [Ab \cos(bu)]^2 & Ab \cos(bu) \\ Ab \cos(bu) & 1 \end{bmatrix} \quad (5)$$

Incidentally, we can check this by calculating the reverse transformation metric:

$$g_{ij} = g_{kl} \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^l}{\partial x^j} \quad (6)$$

For example (using $x = u$):

$$g_{xx} = 1 + [Ab \cos(bx)]^2 - [Ab \cos(bx)]^2 - [Ab \cos(bx)]^2 + [Ab \cos(bx)]^2 \quad (7)$$

$$= 1 \quad (8)$$

Similar calculations give the other components of the rectangular metric.

Now suppose we have an object moving with velocity \mathbf{v} such that $v^x = v$ and $v^y = 0$. In the sinusoidal system, we get

$$v^u = \frac{\partial u}{\partial x^i} v^i \quad (9)$$

$$= v \quad (10)$$

$$v^w = \frac{\partial w}{\partial x^i} v^i \quad (11)$$

$$= -Ab \cos(bu) v \quad (12)$$

$$= -Ab \cos(bvt) v \quad (13)$$

where the last line arises because $u = x = vt$, where t is the time.

The square is

$$v^2 = g'_{ij} v'^i v'^j \quad (14)$$

$$= \left(1 + [Ab \cos(bu)]^2\right) v^2 - [Ab \cos(bu)]^2 v^2 - [Ab \cos(bu)]^2 v^2 + [Ab \cos(bu)]^2 v^2 \quad (15)$$

$$= v^2 \quad (16)$$

Thus the square of the velocity is invariant.

The unit vector \mathbf{e}_u is tangent to the constant- w curves and points in the direction of increasing u . Since the constant- w curves are sine curves, this unit vector starts off horizontal (when $bu = \pi/2$), then slopes downward as bu heads towards $3\pi/2$ at which point it is horizontal again. Then it slopes upwards as bu heads towards $5\pi/2$ and so on. Its magnitude is given by

$$|\mathbf{e}_u| = \sqrt{g_{uu}} = \sqrt{1 + [Ab \cos(bu)]^2}.$$

The other unit vector \mathbf{e}_w is tangent to the constant- u curves, so it always points up, and always has a length of 1. Note that the off-diagonal elements of g'_{ij} are zero when bu is an odd multiple of $\pi/2$, which is where \mathbf{e}_u is horizontal, and thus perpendicular to \mathbf{e}_w , so $g'_{uw} = g'_{wu} = \mathbf{e}_u \cdot \mathbf{e}_w = 0$.

The reason that v^w is not constant, even though the velocity itself is constant is because the unit vector \mathbf{e}_u oscillates in both magnitude and direction.

Since \mathbf{e}_w is a constant, the component multiplying this unit vector (that is, v^w) must also oscillate to compensate for the non-constant u component.

In the rectangular system, the acceleration is zero (since v is constant). In the sinusoidal system, $dv^u/dt = 0$, but $dv^w/dt \neq 0$. Thus the magnitude of a^2 would not be invariant under the transformation of coordinates, so this cannot be the correct way of calculating derivatives in a general coordinate system.

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