METRIC TENSOR: SINUSOIDAL COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 5; Problem 5.5.

As another example of a different coordinate system, we can define a sinusoidal system by introducing coordinates u and w defined as:

$$\begin{aligned} u &= x \quad (1)\\ w &= y-A\sin(bx) \quad (2)\end{aligned}$$

where A and b are constants. If x and y have units of length, then b must have units of length^{-1}. If A has units of length, then u and w both have units of length.

The curves of constant u are just the same as the curves of constant x, that is, vertical lines. The curves of constant w are defined by $y=k+A\sin(bx)$. These are parallel horizontal sine curves displaced vertically by the constant k.

The inverted relations are

$$\begin{aligned} x &= u \quad (3)\\ y &= w+A\sin(bu) \quad (4)\end{aligned}$$

We can work out the metric of the sinusoidal system from the rectangular metric by direct calculation

$$g'_{ij}=g_{kl}\frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}=\left[\begin{array}{cc} 1+[Ab\cos(bu)]^2 & Ab\cos(bu)\\ Ab\cos(bu) & 1\end{array}\right] \quad (5)$$

Incidentally, we can check this by calculating the reverse transformation metric:

$$g_{ij}=g_{kl}\frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j} \quad (6)$$

For example (using $x=u$):

$$g_{xx} = 1 + [Ab\cos(bx)]^2 - [Ab\cos(bx)]^2 - [Ab\cos(bx)]^2 + [Ab\cos(bx)]^2 \tag{7}$$

$$= 1 \tag{8}$$

Similar calculations give the other components of the rectangular metric.

Now suppose we have an object moving with velocity $\mathbf{v}$ such that $v^x = v$ and $v^y = 0$. In the sinusoidal system, we get

$$v^u = \frac{\partial u}{\partial x^i} v^i \tag{9}$$

$$= v \tag{10}$$

$$v^w = \frac{\partial w}{\partial x^i} v^i \tag{11}$$

$$= -Ab\cos(bu)\, v \tag{12}$$

$$= -Ab\cos(bvt)\, v \tag{13}$$

where the last line arises because $u = x = vt$, where t is the time.

The square is

$$v^2 = g'_{ij} v'^i v'^j \tag{14}$$

$$= \left(1 + [Ab\cos(bu)]^2\right) v^2 - [Ab\cos(bu)]^2 v^2 - [Ab\cos(bu)]^2 v^2 + [Ab\cos(bu)]^2 v^2 \tag{15}$$

$$= v^2 \tag{16}$$

Thus the square of the velocity is invariant.

The unit vector $\mathbf{e}_u$ is tangent to the constant-w curves and points in the direction of increasing u. Since the constant-w curves are sine curves, this unit vector starts off horizontal (when $bu = \pi/2$), then slopes downward as bu heads towards $3\pi/2$ at which point it is horizontal again. Then it slopes upwards as bu heads towards $5\pi/2$ and so on. Its magnitude is given by $|\mathbf{e}_u| = \sqrt{g_{uu}} = \sqrt{1 + [Ab\cos(bu)]^2}$.

The other unit vector $\mathbf{e}_w$ is tangent to the constant-u curves, so it always points up, and always has a length of 1. Note that the off-diagonal elements of g'_{ij} are zero when bu is an odd multiple of $\pi/2$, which is where $\mathbf{e}_u$ is horizontal, and thus perpendicular to $\mathbf{e}_w$, so $g'_{uw} = g'_{wu} = \mathbf{e}_u \cdot \mathbf{e}_w = 0$.

The reason that v^w is not constant, even though the velocity itself is constant is because the unit vector $\mathbf{e}_u$ oscillates in both magnitude and direction.

Since $\mathbf{e}_w$ is a constant, the component multiplying this unit vector (that is, v^w) must also oscillate to compensate for the non-constant u component.

In the rectangular system, the acceleration is zero (since v is constant). In the sinusoidal system, $dv^u/dt = 0$, but $dv^w/dt \neq 0$. Thus the magnitude of a^2 would not be invariant under the transformation of coordinates, so this cannot be the correct way of calculating derivatives in a general coordinate system.

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