

## METRIC TENSOR: INVERSE AND RAISING & LOWERING INDICES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.1.

The inverse metric tensor  $g^{ij}$  is defined so that

$$(1) \quad g^{ij} g_{jk} = \delta^i_k$$

If the metric tensor is viewed as a matrix, then this is equivalent to saying  $[g^{ij}] = [g_{ij}]^{-1}$ . The transformation property of  $g^{ij}$  can be worked out by direct calculation, using the transformation of  $g_{ij}$  and the fact that  $\delta^i_k$  is invariant.

$$(2) \quad g'^{ij} g'_{jk} = \delta^i_k$$

$$(3) \quad = g'^{ij} \frac{\partial x^l}{\partial x'^j} \frac{\partial x^m}{\partial x'^k} g_{lm}$$

We can try the transformation

$$(4) \quad g'^{ij} = \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} g^{ab}$$

Substituting, we get

$$(5) \quad g'^{ij} g'_{jk} = \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} g^{ab} \frac{\partial x^l}{\partial x'^j} \frac{\partial x^m}{\partial x'^k} g_{lm}$$

$$(6) \quad = \frac{\partial x^i}{\partial x^a} g^{ab} \delta^l_b \frac{\partial x^m}{\partial x'^k} g_{lm}$$

$$(7) \quad = \frac{\partial x^i}{\partial x^a} g^{al} \frac{\partial x^m}{\partial x'^k} g_{lm}$$

$$(8) \quad = \frac{\partial x^i}{\partial x^a} \frac{\partial x^m}{\partial x'^k} \delta^a_m$$

$$(9) \quad = \frac{\partial x^i}{\partial x^m} \frac{\partial x^m}{\partial x'^k}$$

$$(10) \quad = \delta^i_k$$

On line 2 we used  $\frac{\partial x^j}{\partial x^b} \frac{\partial x^l}{\partial x^j} = \delta^l_b$  and on line 4 we used  $g^{al} g_{lm} = \delta^a_m$ . Thus  $g^{ij}$  is a rank-2 contravariant tensor, and is the inverse of  $g_{ij}$  which is a rank-2 covariant tensor. Since the matrix inverse is unique (basic fact from matrix algebra), we can use the standard techniques of matrix algebra to calculate the inverse.

In rectangular coordinates,  $g^{ij} = g_{ij}$  since the metric is diagonal with all diagonal elements equal to 1. In polar coordinates in 2-d,

$$(11) \quad g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

so the inverse is

$$(12) \quad g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$$

A contravariant vector  $v^i$  can be *lowered* (converted to a covariant vector) by multiplying by  $g_{ij}$ :

$$(13) \quad v_i = g_{ij} v^j$$

The covariant vector can be converted back into a contravariant vector by *raising* its index:

$$(14) \quad g^{ij} v_j = g^{ij} g_{jk} v^k$$

$$(15) \quad = \delta^i_k v^k$$

$$(16) \quad = v^i$$

If we start with a vector  $v^i$  in rectangular coordinates, we can convert it to polar coordinates:

$$(17) \quad v^r = v^x \cos \theta + v^y \sin \theta$$

$$(18) \quad v^\theta = -v^x \frac{\sin \theta}{r} + v^y \frac{\cos \theta}{r}$$

We can lower these components by multiplying by  $g_{ij}$

$$(19) \quad v_r = v^x \cos \theta + v^y \sin \theta$$

$$(20) \quad v_\theta = r^2 \left( -v^x \frac{\sin \theta}{r} + v^y \frac{\cos \theta}{r} \right)$$

$$(21) \quad = r(-v^x \sin \theta + v^y \cos \theta)$$

The square magnitude is

$$(22) \quad v^i v_i = v^r v_r + v^\theta v_\theta$$

$$(23) \quad = (v^x \cos \theta + v^y \sin \theta)^2 + (-v^x \sin \theta + v^y \cos \theta)^2$$

$$(24) \quad = (v^x)^2 + (v^y)^2$$

(No implied sum on the RHS in line 1.)

#### PINGBACKS

Pingback: Gradient as covector: example in 2-d

Pingback: Tensor trace

Pingback: Metric tensor: trace