METRIC TENSOR: INVERSE AND RAISING & LOWERING INDICES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 6; Problem 6.1.

The inverse metric tensor g^{ij} is defined so that

$$g^{ij}g_{jk}=\delta^{i}{}_{k} \tag{0.1}$$

If the metric tensor is viewed as a matrix, then this is equivalent to saying $\left[g^{ij}\right]=\left[g_{ij}\right]^{-1}$. The transformation property of g^{ij} can be worked out by direct calculation, using the transformation of g_{ij} and the fact that $\delta^{i}{}_{k}$ is invariant.

$$\begin{aligned}
g^{\prime ij}g_{jk}^{\prime} &= \delta^{i}{}_{k} && (0.2)\\
&= g^{\prime ij}\frac{\partial x^{l}}{\partial x^{\prime j}}\frac{\partial x^{m}}{\partial x^{\prime k}}g_{lm} && (0.3)
\end{aligned}$$

We can try the transformation

$$g^{\prime ij}=\frac{\partial x^{\prime i}}{\partial x^{a}}\frac{\partial x^{\prime j}}{\partial x^{b}}g^{ab} \tag{0.4}$$

Substituting, we get

$$\begin{aligned}
g^{\prime ij}g_{jk}^{\prime} &= \frac{\partial x^{\prime i}}{\partial x^{a}}\frac{\partial x^{\prime j}}{\partial x^{b}}g^{ab}\frac{\partial x^{l}}{\partial x^{\prime j}}\frac{\partial x^{m}}{\partial x^{\prime k}}g_{lm} && (0.5)\\
&= \frac{\partial x^{\prime i}}{\partial x^{a}}g^{ab}\delta^{l}{}_{b}\frac{\partial x^{m}}{\partial x^{\prime k}}g_{lm} && (0.6)\\
&= \frac{\partial x^{\prime i}}{\partial x^{a}}g^{al}\frac{\partial x^{m}}{\partial x^{\prime k}}g_{lm} && (0.7)\\
&= \frac{\partial x^{\prime i}}{\partial x^{a}}\frac{\partial x^{m}}{\partial x^{\prime k}}\delta^{a}{}_{m} && (0.8)\\
&= \frac{\partial x^{\prime i}}{\partial x^{m}}\frac{\partial x^{m}}{\partial x^{\prime k}} && (0.9)\\
&= \delta^{i}{}_{k} && (0.10)
\end{aligned}$$

On line 2 we used $\frac{\partial x'^j}{\partial x^b}\frac{\partial x^l}{\partial x'^j} = \delta^l{}_b$ and on line 4 we used $g^{al}g_{lm} = \delta^a_m$. Thus g^{ij} is a rank-2 contravariant tensor, and is the inverse of g_{ij} which is a rank-2 covariant tensor. Since the matrix inverse is unique (basic fact from matrix algebra), we can use the standard techniques of matrix algebra to calculate the inverse.

In rectangular coordinates, $g^{ij} = g_{ij}$ since the metric is diagonal with all diagonal elements equal to 1. In polar coordinates in 2-d,

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{0.11}$$

so the inverse is

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \tag{0.12}$$

A contravariant vector v^i can be *lowered* (converted to a covariant vector) by multiplying by g_{ij}:

$$v_i = g_{ij}v^i \tag{0.13}$$

The covariant vector can be converted back into a contravariant vector by *raising* its index:

$$\begin{aligned} g^{ij}v_j &= g^{ij}g_{jk}v^k && (0.14)\\ &= \delta^i{}_k v^k && (0.15)\\ &= v^i && (0.16) \end{aligned}$$

If we start with a vector v^i in rectangular coordinates, we can convert it to polar coordinates:

$$\begin{aligned} v^r &= v^x\cos\theta + v^y\sin\theta && (0.17)\\ v^\theta &= -v^x\frac{\sin\theta}{r} + v^y\frac{\cos\theta}{r} && (0.18) \end{aligned}$$

We can lower these components by multiplying by g_{ij}

$$\begin{aligned} v_r &= v^x\cos\theta + v^y\sin\theta && (0.19)\\ v_\theta &= r^2\left(-v^x\frac{\sin\theta}{r} + v^y\frac{\cos\theta}{r}\right) && (0.20)\\ &= r\left(-v^x\sin\theta + v^y\cos\theta\right) && (0.21) \end{aligned}$$

The square magnitude is

$$v^i v_i = v^r v_r + v^\theta v_\theta \tag{0.22}$$
$$= (v^x \cos\theta + v^y \sin\theta)^2 + (-v^x \sin\theta + v^y \cos\theta)^2 \tag{0.23}$$
$$= (v^x)^2 + (v^y)^2 \tag{0.24}$$

(No implied sum on the RHS in line 1.)

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